3. Since, $3x-4y+4=0$ and $3x-4y-\frac{7}{2}=0$ are two parallel tangents.

Thus distance between them is diameter of circle

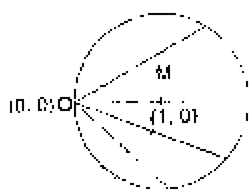
$$\Rightarrow \text{diameter} = \frac{\left|4+\frac{7}{2}\right|}{\sqrt{3^2+4^2}} = \frac{15}{2.5} = \frac{3}{2}$$

or radius $=\frac{3}{4}$

4. For the equation of circle $x^2+y^2-2x=0$. Let the mid point of chords be (h, k)

$\therefore$ Equation of chord bisected at the point is $S_1=T$

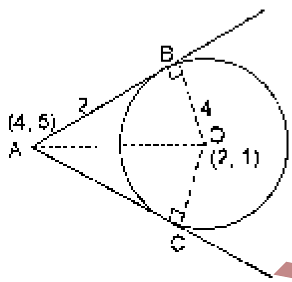
$h^2+k^2-2h = xh+yk-(x+h)$ which passes through $(0, 0)$



$\Rightarrow h^2+k^2-2h=-h$ is required equation of locus

or $x^2+y^2-x=0$ locus of mid point of chords.

5. Here, to find area of quadrilateral $ABOC$



$\therefore$ Area of quadrilateral $ABOC = 2$ area of ΔABO

$$=2\cdot\frac{1}{2}(AB)(OB)$$

$$=2\cdot 4 = 8 \text{ square units}$$

6. Equation of straight line passing through intersection of two circles C_1 and C_2 is $(S_1-S_2)=0$

$$\Rightarrow \frac{20}{3}x-2y-12=0$$

or $10x-3y-18=0$

7. $(x+2)^2+(y-3)^2=4$

let if $M(h, k)$, where B is mid point of A and M.

$$\Rightarrow B\left(\frac{h}{2}, \frac{k+3}{2}\right)$$

But AB is the chord of circle $x^2+4x+(y-3)^2=0$

Thus, B must satisfy above equation

$$i.e., \frac{h^2}{4}+\frac{4h}{2}+\left\{\frac{1}{2}(k+3)-3\right\}^2=0$$

or $h^2+k^2+8h-6k+9=0$

$\therefore$ locus of M is the circle

$$x^2+y^2+8x-6y+9=0$$

8. Area of Δ formed by the tangents from the point (h, k) to the circle $x^2+y^2=a^2$ and their chord of contact

$$=a\frac{(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$$

Thus, area of Δ formed by tangents from (4, 3) to the circle $x^2+y^2=9$ and their chord of contact

$$=\frac{3(4^2+3^2-9)^{3/2}}{4^2+3^2}$$

$$=\frac{3(16+9-9)^{3/2}}{25}$$

$$=\frac{3.(64)}{25}=\frac{192}{25} \text{ square units}$$

9. $C_1: x^2+y^2=16$

$C_2: (x-h)^2+(y-k)^2=25$

Common chords by $S_1-S_2=0$ is

$$2hx+2ky=(h^2+k^2-9)$$

$\therefore$ Its slope $=-\frac{h}{k}=\frac{3}{4}$, given

If p be the length of perpendicular on it from the centre (0, 0) of C_1 of radius 4, then $p=\frac{h^2+k^2-9}{\sqrt{4h^2+4k^2}}$. Also the length of the chord is

$$2\sqrt{r^2-p^2}=2\sqrt{4^2-p^2}$$

The chord will be of maximum length, if $p=0$ or $h^2+k^2-9=0$ or $h^2+\frac{16}{9}h^2=9$

or $h=\pm\frac{9}{5}$

$\therefore$ $k=\mp\frac{12}{5}$

Centres are $\left(\frac{9}{5}, -\frac{12}{5}\right)$ and $\left(-\frac{9}{5}, \frac{12}{5}\right)$

10. As the point of intersection of the coordinate axes with the line $\lambda x-y+1=0$ and $x-2y+3=0$ forms the circle;

$\therefore$ $(\lambda x-y+1)(x-2y+3)=0$

represents a circle if, coefficient of x^2 = coefficient of y^2 and coefficient of $xy=0$

$\Rightarrow$ $\lambda=2$ and $-2\lambda-1=0$

$\Rightarrow$ $\lambda=2$ or $\lambda=-\frac{1}{2}$

11. $4x^2+4y^2-12x+4y+1=0$ (given)

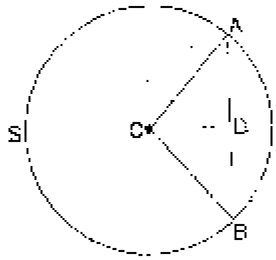
$\Rightarrow$ $x^2+y^2-3x+y+1/4=0$

centre of this circle is $\left(\frac{3}{2}, -\frac{1}{2}\right)$ and radius

$$= \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 - \frac{1}{4}}$$

$$= \sqrt{\frac{9}{4} + \frac{1}{4} - \frac{1}{4}}$$

$$= \frac{3}{2}$$



Again let S is circle with centre at C and AB is given chord and AD subtend angle $2\pi/3$ at the centre and D be the mid-point of AB and let its coordinates are (h, k).

Now $\quad \angle DCA = \frac{1}{2}(\angle BCA) = \frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{3}$

Again $\quad \frac{DA}{\sin \pi/3} = \frac{CA}{\sin \pi/2}$

$\Rightarrow \quad DA = CA \sin \pi/3 = \frac{3}{2} \cdot \frac{\sqrt{3}}{2}$

Now, in ΔACD

$$CD^2 = CA^2 - AD^2$$

$\Rightarrow \quad = \frac{9}{4} - \frac{27}{16} = \frac{9}{16}$

But $\quad CD^2 = (h - 3/2)^2 + (k + 1/2)^2$

$\Rightarrow \quad (h - 3/2)^2 + (k + 1/2)^2 = \frac{9}{16}$

On generalising, we get

$$\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{9}{16}$$

$\Rightarrow \quad 16x^2 + 16y^2 - 48x + 16y + 31 = 0$

12. In an equilateral triangle the radius of incircle

$= \frac{1}{3} \times$ median of the triangle

$= \frac{1}{3}\sqrt{a^2 - a^2/4} = \frac{1}{3}\sqrt{\frac{4a^2 - a^2}{4}} = \sqrt{\frac{3a^2}{32}} = \frac{a}{2\sqrt{3}}$

Therefore, area of the square inscribed in this circle.

$= 2(\text{radius of circle})^2$

$= \frac{2a^2}{43} = \frac{a^2}{6}$ sq. units.

B TRUE / FALSE

1. As, centre of circle is $(3, -1)$ which lies on $x + 3y = 0$

C OBJECTIVE (ONLY ONE OPTION)

1. The required equation of circle is, $C_1 + \lambda(C_2 - C_1) = 0$

$\Rightarrow (x^2 + y^2 - 6) - \lambda(-6x + 14) = 0$ passing through $(1, 1)$

13. Equation of any circle passing through the point of intersection of $x^2 + y^2 - 2x = 0$ and $y - x$ is

$$(x^2 + y^2 - 2x) + \lambda(y - x) = 0$$

$\Rightarrow \quad x^2 + y^2 - (2 + \lambda)x + \lambda y = 0$

Its centre is $\left(\frac{2 + \lambda}{2}, \frac{-\lambda}{2}\right)$

$\Rightarrow$ For AB to be the diameter of the required circle the centre must be on AB. That is

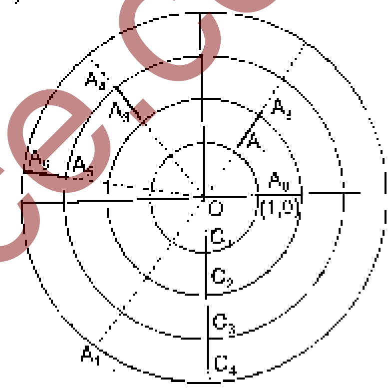
$$2 + \lambda = -\lambda \quad \Rightarrow \quad \lambda = -1$$

Therefore, equation of the required circle is

$$x^2 + y^2 - (2 - 1)x - 1 \cdot y = 0$$

$\Rightarrow \quad x^2 + y^2 - x - y = 0$

14. It is given that C_1 has centre $(0, 0)$ and radius 1. Similarly, C_2 has centre $(0, 0)$ and radius 2, and C_k has centre $(0, 0)$ and radius k.



Now particle starts it motion from $(1, 0)$ and moves 1 radian on first circle then particle shifts from C_1 to C_2. After that particle moves 1 radian on C_2 and then particle shifts from C_2 to C_3. Similarly, particle move on n circles.

Now $n \geq 2\pi$ because particle crosses the x-axis for the first time on C_n, then n is least positive integer.

Therefore, $n = 7$ is the answer.

15. A point on the line $2x + y = 4$ is of the form $(h, 4 - 2h)$.

Equation of the chord of contact is $T = 0$ i.e.,

$$hx + (4 - 2h)y = 1$$

or $\quad (4y - 1) + h(x - 2y) = 0$

This line passes through the point of intersection of $4y - 1 = 0$ and $x - 2y = 0$ i.e., through the point $\left(\frac{1}{2}, \frac{1}{4}\right)$

$\Rightarrow x + 3y = 0$ is diameter of $x^2 + y^2 - 6x - 2y - 0$

$\Rightarrow \quad -4 + \lambda(8) = 0 \quad$ or $\quad \lambda = \frac{1}{2}$

$\therefore$ Required equation of circle is,

$$x^2 + y^2 - 6 - 3x + 7 = 0$$

or $$x^2 + y^2 - 3x + 1 = 0$$

2. The required equation of circle is,

$$(x^2 + y^2 + 13x - 3y) + \lambda\left(11x + \frac{1}{2}y + \frac{25}{2}\right) = 0 \quad ...(1)$$

passing through (1, 1)

$\Rightarrow \quad 12 + \lambda(24) = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{2}$

putting in (1), we get

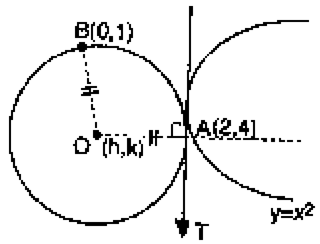
$$x^2 + y^2 + 13x - 3y - \frac{11}{2}x - \frac{1}{4}y - \frac{25}{4} = 0$$

or $$4x^2 + 4y^2 + 52x - 12y - 22x - y - 25 = 0$$

or $$4x^2 + 4y^2 + 30x - 13y - 25 = 0$$

3. Let centre of circle be (h, k)

So that, $OA^2 = OB^2$



and (slope of OA) (slope of tangent at A) = -1

$\Rightarrow \quad h^2 + (k-1)^2 = (h-2)^2 + (k-4)^2$

or $$4h + 6k - 19 = 0 \quad ...(1)$$

also, slope of $OA = \frac{k-4}{h-2}$ and slope of tangent at (2, 4) to $y = x^2$ is 4

$\therefore \quad \frac{k-4}{h-2} \cdot 4 = -1$

or $\quad 4k - 16 = -h + 2$

$$h + 4k = 18 \quad ...(2)$$

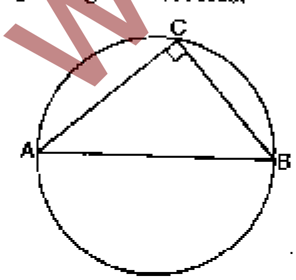
solving (1) and (2), we get

$$k = \frac{53}{10} \text{ and } h = -\frac{16}{5}$$

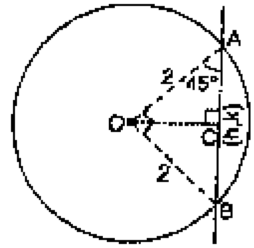
$\therefore$ Centre co-ordinates are $\left(-\frac{16}{5}, \frac{53}{10}\right)$

4. Clearly, $\angle C = 90°$ as angle in semicircle is right angled, now area of Δ is maximum when $AC = BC$.

i.e., Δ is right angled isosceles.



5. As, we have to find locus of mid-point of chord and we know perpendicular from centre bisects the chord.



$\therefore \quad \angle OAC = 45°$

or $\quad \frac{OC}{OA} = \sin 45° \Rightarrow OC = \frac{2}{\sqrt{2}}$

$\therefore \quad \sqrt{h^2 + k^2} = OC^2$

or $x^2 + y^2 = 2$ is required equation of locus of mid-point of chord subtending right angle at centre.

6. Let $x^2 + y^2 + 2gx + 2fy + c = 0$, cuts $x^2 + y^2 = k^2$ orthogonally.

$\Rightarrow \quad 2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$\Rightarrow \quad -2g0 - 2f \cdot 0 = c - k^2$

or $$c = k^2 \quad ...(1)$$

Also, $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through (a, b)

$\therefore \quad a^2 + b^2 + 2ga + 2fb + c = 0 \quad ...(2)$

$\Rightarrow$ Required equation of locus of centre is,

$$-2ax - 2by + a^2 + b^2 + k^2 = 0$$

or $$2ax + 2by - (a^2 + b^2 + k^2) = 0$$

7. As, if the two circles intersect in two distinct points

$\Rightarrow$ distance between centres lies between $|r_1 - r_2|$ and $|r_1 + r_2|$.

i.e., $|r - 3| < \sqrt{(4-1)^2 + (+1 + 3)^2} < |r + 3|$

$\Rightarrow \quad |r - 3| < 5 < |r + 3|$

$\therefore \quad r < 8$ or $r > 2$

$\therefore \quad 2 < r < 8$

8. Since, $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle

$\Rightarrow$ Their point of intersection is centre *i.e.*, (1, −1) centre and area, $\pi r^2 = 154$

or $\quad r^2 = 154 \times \frac{7}{22}$

$\Rightarrow \quad r = 7$

$\therefore$ Required equation of circle is,

$$(x-1)^2 + (y+1)^2 = 7^2$$

or $$x^2 + y^2 - 2x + 2y = 47$$

9. Let (h, k) be the centre of the required circle. Then

$$\sqrt{(h-0)^2 + (k-0)^2} = \sqrt{(h-1)^2 + (k-0)^2}$$

$\Rightarrow \quad \sqrt{h^2 + k^2} = \sqrt{h^2 - 2h + 1 + k^2}$

$\Rightarrow \quad h^2 + k^2 = h^2 - 2h + 1 + k^2$

$\Rightarrow \quad -2h + 1 = 0$

$\Rightarrow \quad h = 1/2$

Now, (0, 0) and (1, 0) lie inside the circle $x^2+y^2=9$. Therefore, the required circle can touch the given circle internally.

i. e., $\quad C_1C_2 = r_1 - r_2$

$\Rightarrow \quad \sqrt{h^2+k^2} = 3 - \sqrt{h^2+k^2}$

$\Rightarrow \quad 2\sqrt{h^2+k^2} = 3$

$\Rightarrow \quad 2\sqrt{\frac{1}{4}+k^2} = 3$

$\Rightarrow \quad \sqrt{\frac{1}{4}+k^2} = \frac{3}{2}$

$\Rightarrow \quad \frac{1}{4}+k^2 = \frac{9}{4}$

$\Rightarrow \quad k^2 = 2$

$\Rightarrow \quad k = \pm\sqrt{2}$

therefore, (d) is the answer.

10. Let (h, k) be the centre of the circle which touches the circle $x^2+y^2-6x-6y+14=0$ and y-axis.

Now, $x^2+y^2+2(-3)x-2(-3)y+14=0$, the centre of this circle is (3, 3)

and radius is $\sqrt{3^2+3^2-14} = \sqrt{9+9-14}$

$= \sqrt{18-14} = \sqrt{4} = 2$

Since, the circle touches y-axis, the distance from its centre to y-axis must be equal to its radius, therefore its radius is h. Again the circles meet externally, therefore the distance between two centres = sum of the radii of the two circles.

Hence, $\quad (h-3)^2+(k-3)^2 = (2+h)^2$

i.e. $\quad h^2+9-6h+k^2+9-6k = 4+h^2+4h$

i.e. $\quad k^2-10h-6k+14=0$

Thus, the locus of (h, k) is

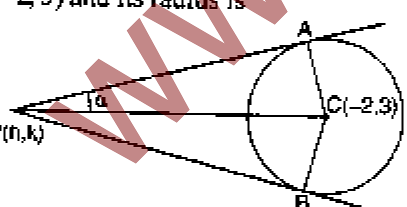
$$y^2-10x-6y+14=0$$

Therefore, (d) is the answer.

11. Centre of the circle

$$x^2+y^2+4x-6y+9\sin^2\alpha+13\cos^2\alpha=0$$

is $C(-2, 3)$ and its radius is



$\sqrt{(-2)^2+(+3)^2-9\sin^2\alpha-13\cos^2\alpha}$

$= \sqrt{4+9-9\sin^2\alpha-13\cos^2\alpha}$

$= \sqrt{13-13\cos^2\alpha-9\sin^2\alpha}$

$= \sqrt{13(1-\cos^2\alpha)-9\sin^2\alpha}$

$= \sqrt{13\sin^2\alpha-9\sin^2\alpha}$

$= \sqrt{4\sin^2\alpha} = 2\sin\alpha$

Let (h, k) be any point P and $\angle APC = \alpha$, $\angle PAC = \pi/2$

That is, triangle APC is a right triangle.

Thus, $\quad \sin\alpha = \frac{AC}{PC} = \frac{2\sin\alpha}{\sqrt{(h+2)^2+(k-3)^2}}$

$\Rightarrow \quad \sqrt{(h+2)^2+(k-3)^2} = 2$

$\Rightarrow \quad (h+2)^2+(k-3)^2 = 4$

$\Rightarrow \quad h^2-4+4h+k^2+9-6k = 4$

$\Rightarrow \quad h^2+k^2+4h-6k+9=0$

Thus, required equation of the locus is

$$x^2+y^2+4x-6y+9=0$$

12. $x^2+y^2=4$ (given)

Centre $= C_1 \equiv (0, 0)$ and $R_1 = 2$

Again $x^2+y^2-6x-8y-24=0$, then $C_2 \equiv (3, 4)$

and $\quad R_2 = 7$

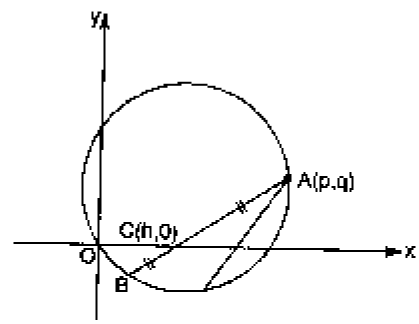
Again $\quad C_1C_2 = 5 = R_2 - R_1$

Therefore, the given circles touch internally such that they can have just one common tangents at the point of contact

Therefore, (b) is the answer.

13. **Imp. Note :** In solving a line and a circle there oftengenerate a quadratic equation and further we have to apply condition of Discriminant so question convert from coordinate to quadratic equation.

From equation of circle it is clear that circle passes through origin. Let AB is chord of the circle.



$A \equiv (p, q)$. C is mid-point and Co-ordinate of C is $(h, 0)$

Then coordinates of B are $(-p+2h, -q)$.

and B lies on the circle

$x^2+y^2 = px - qy$, we have

$$(-p+2h)^2+(-q)^2 = p(-p+2h)+q(-q)$$

$\Rightarrow \quad p^2+4h^2-4ph+q^2 = -p^2+2ph-q^2$

$\Rightarrow \quad 2p^2+2q^2-6ph+4h^2=0$

$\Rightarrow \quad 2h^2-3ph+p^2+q^2=0 \quad \ldots(1)$

There are given two distinct chords which are bisected at x-axis then, there will be two distinct values of h satisfying (1).

So discriminant of this quadratic equation must be > 0.

$\Rightarrow \quad D > 0$

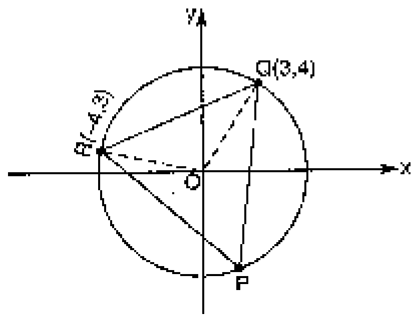
$\Rightarrow \quad (-3p)^2 - 4\cdot 2(p^2 + q^2) > 0$

$\Rightarrow \quad 9p^2 - 8p^2 - 8q^2 > 0$

$\Rightarrow \quad p^2 - 8q^2 > 0$

$\Rightarrow \quad p^2 > 8q^2$ Therefore, (d) is the answer.

14. O is the point at centre and P is the point at circumference. Therefore, angle QOR is double the angle QPR. So it is sufficient to find the angle QOR.



Now slope of $OQ = 4/3$

slope of $OR = -3/4$

again $m_1 m_2 = -1$

therefore, $\angle QOR = 90°$

which implies that $\angle QPR = 45°$

Therefore, (c) is the answer.

15. $2g_1 g_2 + 2f_1 f_2 = c_1 + c_2$ (Formula for orthogonal intersection)

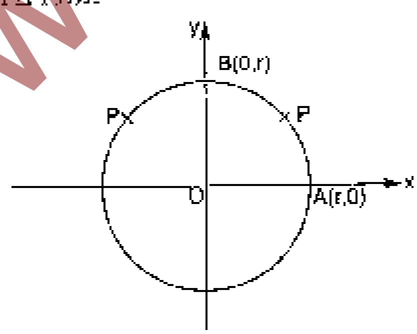
$$2(1)(0) + 2(k)(k) = 6 + k$$

$\Rightarrow \quad 2k^2 - k - 6 = 0$

$\Rightarrow \quad k = -\dfrac{3}{2}, 2$

Therefore, (a) is the answer.

16. Choosing OA as x-axis, $A = (r, 0)$, $B = (0, r)$ and any point P on the circle is $(r\cos\theta, r\sin\theta)$. If (x, y) is the centroid of ΔPAB.



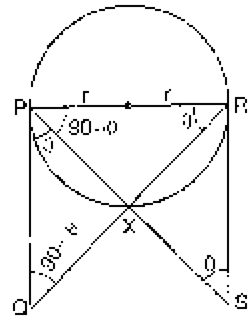
$$3x = r\cos\theta + r + 0$$

$$3y = r\sin\theta + 0 + r$$

$\therefore \quad (3x - r)^2 + (3y - r)^2 = r^2$

Therefore, locus of P is a circle. So (b) is the answer.

17. From figure it is clear that ΔPRQ and ΔRSP are similar.



$$\frac{PR}{RS} = \frac{PQ}{RP}$$

$\Rightarrow \quad PR^2 = PQ \cdot RS$

$\Rightarrow \quad PR = \sqrt{PQ \cdot RS}$

$\Rightarrow \quad 2r = \sqrt{PQ \cdot RS}$

Therefore, (a) is the answer.

18. The line $5x - 2y + 6 = 0$ meets the y-axis at the point $(0, 3)$ and therefore the tangent has to pass through the point $(0, 3)$ and required length is therefore,

$$= \sqrt{x_1^2 + y_1^2 + 6x_1 + 6y_1 - 2}$$

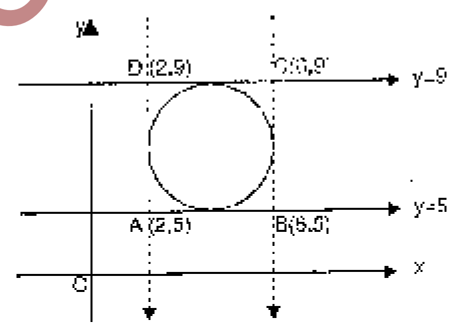
$$= \sqrt{0^2 + 3^2 + 6(0) - 6(3) - 2}$$

$$= \sqrt{25} = 5$$

19. Given the circle is inscribed in square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$

$\Rightarrow \quad x = 6$ and $x = 2$, $y = 5$ and $y = 9$

which could be plotted as:



where $ABCD$ clearly forms a square.

$\therefore$ Centre of inscribed circle

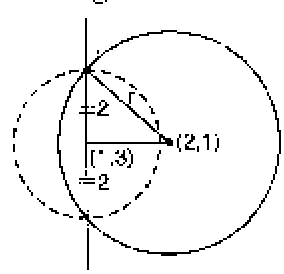
= point of intersection of diagonals

= mid point of AC or BD

$$= \left(\left(\frac{2+6}{2}\right), \left(\frac{5+9}{2}\right) = (4, 7)\right)$$

$\Rightarrow$ Centre of inscribed circle is (4, 7)

20. Clearly from the figure the radius of bigger circle



$$r^2 = 2^2 + [(2-1)^2 + (1-3)^2]$$

$$r^2 = 9 \text{ or } r = 3$$

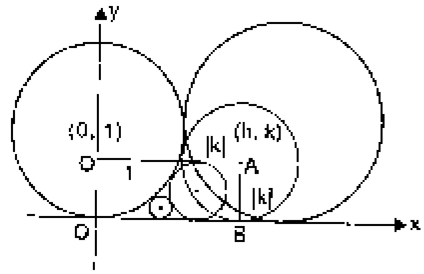
21. Let the locus of centre of circle be (h, k) touching $(y-1)^2 + x^2 = 1$ and x-axis shown as :

Clearly from figure,

distance between O and A is always $1+|k|$.

i.e., $\sqrt{(h-0)^2+(k-1)^2} = 1+|k|$,

squaring both sides, we get



$$h^2+k^2-2k+1 = 1+k^2+2|k|$$

$\Rightarrow$ $h^2 = 2|k|+2k$

or $x^2 = 2|y|+2y$

where $|y| = \begin{cases} y & y \ge 0 \\ -y & y<0 \end{cases}$

$\therefore$ $x^2 = 2y+2y,\ y\ge 0$

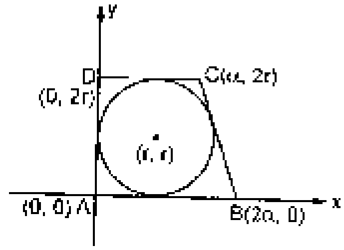
and $x^2 = -2y+2y,\ y<0$

$\Rightarrow$ $x^2 = 4y$ when $y \ge 0$

and $x^2 = 0$ when $y>0$

$\therefore \{(x, y): x^2 = 4y \text{ when } y>0\} \cup \{(0, y): y<0\}$

22. $18 = \frac{1}{2}(3\alpha)(2r) \Rightarrow \alpha r = 6$



Line, $y = -\frac{2r}{\alpha}(x-2\alpha)$ is tangent to circle

$$(x-r)^2+(y-r)^2 = r^2$$

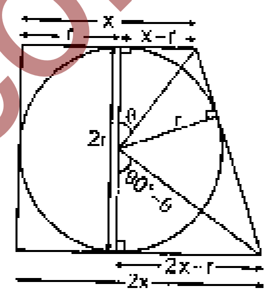
$2\alpha = 3r$ and $\alpha r = 6$

$r = 2$.

Alter :

$\frac{1}{2}(x+2x)\times 2r = 18$

$xr = 6$ ... (i)



$\tan\theta = \frac{x-r}{r}$

$\tan(90°-\theta) = \frac{2x-r}{r}$

$\frac{x-r}{r} = \frac{r}{2x-r} \Rightarrow x(2x-3r) = 0$

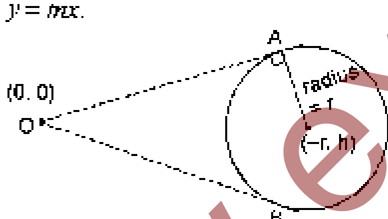
$x = \frac{3r}{2}$ ... (ii)

From Eqs. (i) and (ii), we get

$r = 2$.

D OBJECTIVE (MORE THAN ONE OPTION)

1. Since, tangents are drawn from origin. So let equation of tangent be $y = mx$.



$\Rightarrow$ length of ⊥'r from origin = radius

$\Rightarrow$ $\frac{|mr+h|}{\sqrt{m^2+1}} = r$

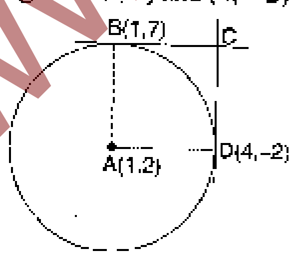
$\Rightarrow$ $m^2r^2+h^2+2mrh = r^2(m^2+1)$

$\Rightarrow$ $m = \left|\frac{r^2-h^2}{2rh}\right|$

$\therefore$ Equation of tangents are, $y = \left|\frac{r^2-h^2}{2rh}\right| x$

E SUBJECTIVE QUESTIONS

1. Equation of tangent at (1, 7) and (4, −2) are



$$x+7y-(x+1)-2(y+7)-20 = 0$$

or $5y = 35$ or $y = 7$

and $4x-2y-(x+4)-2(y-2)-20 = 0$

or $3x-4y = 20$

$\therefore$ point C is (16, 7).

$\Rightarrow$ $A(1, 2), B(1, 7), C(16, 7), D(4, -2)$

Hence, area of quadrilateral $ABCD$

$$= \frac{1}{2}\begin{vmatrix} 1 & 2 \\ 1 & 7 \\ 16 & 7 \\ 4 & -2 \\ 1 & 2 \end{vmatrix}$$

$$= \frac{1}{2}|\{(7+7-32+8)-(2+112+28-2)\}|$$

$$= \frac{1}{2} \cdot 10 \cdot 140 = 75 \text{ sq. units.}$$

2. Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy - c = 0 \quad \ldots(1)$$

It passes through $(-4, 3)$. Therefore,

$$25 - 8g + 6f + c = 0 \quad \ldots(2)$$

Since, circle (1) touches the line $x + y - 2 = 0$ and $x - y - 2 = 0$. Therefore,

$$\left|\frac{-g - f - 2}{\sqrt{2}}\right| = \left|\frac{-g + f - 2}{\sqrt{2}}\right| = \sqrt{g^2 + f^2 - c} \quad \ldots(3)$$

Now, $\left|\frac{-g - f - 2}{\sqrt{2}}\right| = \left|\frac{-g - f - 2}{\sqrt{2}}\right|$

$\Rightarrow \quad -g - f - 2 = \pm(-g + f - 2)$

$\Rightarrow \quad -g - f - 2 = -g + f - 2$

or $\quad -g - f - 2 = g - f + 2$

$\Rightarrow \quad f = 0$ or $g = -2$

Case I : When $f = 0$

from (3), we get

$$\Rightarrow \quad \left|\frac{-g - 2}{\sqrt{2}}\right| = \sqrt{g^2 - c}$$

$$\Rightarrow \quad (g + 2)^2 = 2(g^2 - c)$$

$$\Rightarrow \quad g^2 - 4g - 4 - 2c = 0 \quad \ldots(4)$$

putting $f = 0$ in (2), we get

$$25 - 8g + c = 0 \quad \ldots(5)$$

Eliminating c between (4) and (5), we get

$$g^2 - 20g + 46 = 0$$

$$\Rightarrow \quad g = 10 \pm 3\sqrt{6} \quad \text{and} \quad c = 55 \pm 24\sqrt{6}$$

Substituting the values of g, f and c in (1), we get

$$x^2 + y^2 \pm (2(10 \pm 3\sqrt{6}))x + (55 \pm 24\sqrt{6}) = 0$$

as the equation of the required circle.

Case II : When $g = -2$

from (3), we get

$$\Rightarrow \quad f^2 = 2(4 + f^2 - c)$$

$$\Rightarrow \quad f^2 - 2c + 8 = 0 \quad \ldots(6)$$

putting $g = -2$ in (2), we get

$$c = -6f - 41$$

Substituting 'c' in (6), we get

$$f^2 + 12f + 90 = 0$$

This equation gives imaginary values of f.

Thus, there is no circle in this case.

Hence, the required equation of the circles are,

$$x^2 + y^2 \pm 2(10 + 3\sqrt{6})x + (55 \pm 24\sqrt{6}) = 0$$

3. Circle is $x^2 + y^2 = r^2$

Equation of chord whose mid-point is given is,

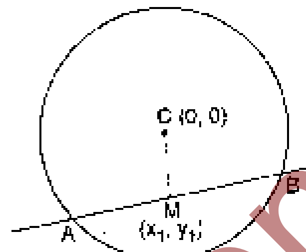
$$S_1 = T$$

$$\Rightarrow \quad xx_1 + yy_1 - r^2 = x_1^2 + y_1^2 - r^2$$

(as it passes through (h, k))

$$hx_1 + ky_1 = x_1^2 + y_1^2$$

$\therefore$ locus of (x_1, y_1) is,



$$x^2 + y^2 = hx + ky$$

Alter :

let M be the mid-point of chord AB.

$\Rightarrow \quad CM \perp MP$

$\Rightarrow \quad$ (slope of CM) (slope of MP) $= -1$

$$\Rightarrow \quad \frac{y}{x} \cdot \frac{k - y}{h - x} = -1$$

$$\Rightarrow \quad ky - y^2 = -hx + x^2$$

or $x^2 + y^2 = hx + ky$ is required locus.

4. Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of points A and B respectively.

It is given that x_1, x_2 are roots of $x^2 + 2ax - b^2 = 0$

$$\Rightarrow \quad x_1 + x_2 = -2a \quad \text{and} \quad x_1x_2 = -b^2 \quad \ldots(1)$$

also, y_1 and y_2 are roots of $y^2 + 2py - q^2 = 0$

$$\Rightarrow \quad y_1 + y_2 = -2p \quad \text{and} \quad y_1y_2 = -q^2 \quad \ldots(2)$$

$\therefore$ The equation of circle with AB as diameter is,

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow \quad x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + (x_1x_2 + y_1y_2) = 0$$

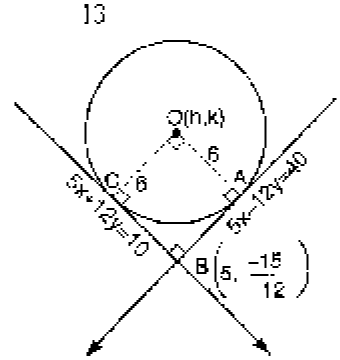
$$\Rightarrow \quad x^2 + y^2 + 2ax + 2py - (b^2 + q^2) = 0$$

5. Since, $5x + 12y - 10 = 0$ and $5x - 12y - 40 = 0$ are both perpendicular tangents to the circle, C_1

$\therefore OABC$ forms a square

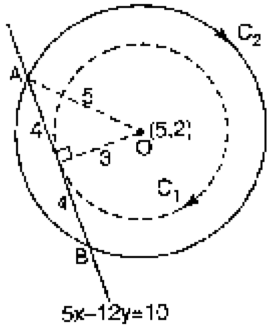
Let the centre coordinates be (h, k) where, $OC = 6$, $OA = 6$ and $OB = 6\sqrt{2}$.

$$\Rightarrow \quad \frac{(5h + 12k - 10)}{13} = 3$$



$\Rightarrow$ $5h+12k-10=\pm 39$ and $5h-12k-40=\pm 39$ on solving above equations. The co-ordinates which lie in I quadrant are (5, 2)

$\therefore$ Centre for C_1 (5, 2)

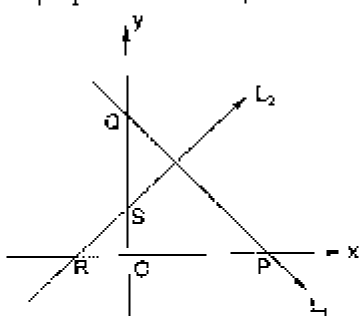


To obtain equation of circle concentric with C_1 and making an intercept of length 8 on $5x+12y=10$ and $5x-12y=40$

$\Rightarrow$ required equation of circle C_2 has centre (5, 2) and radius 5

$$\Rightarrow \quad (x-5)^2+(y-2)^2=5^2$$

6. Let the equation of L_1 be $x\cos\alpha + y\sin\alpha = p_1$.
Then any line perpendicular to L_1 is,



$$x\sin\alpha - y\cos\alpha = p_2$$

where p_2 is a variable.
Then L_1 meets x-axis at $P(p_1\sec\alpha, 0)$ and y-axis at $Q(0, p_1\operatorname{cosec}\alpha)$.
Similarly L_2 meets x-axis at $R(p_2\operatorname{cosec}\alpha, 0)$ and y-axis at $S(0, -p_2\sec\alpha)$.
Now equation of PS is,

$$\frac{x}{p_1\sec\alpha} + \frac{y}{-p_2\sec\alpha} = 1$$

$$\Rightarrow \quad \frac{x}{p_1} - \frac{y}{p_2} = \sec\alpha \qquad \ldots(1)$$

Similarly equation of QR is,

$$\frac{x}{p_2\operatorname{cosec}\alpha} + \frac{y}{p_1\operatorname{cosec}\alpha} = 1$$

$$\Rightarrow \quad \frac{x}{p_2} + \frac{y}{p_1} = \operatorname{cosec}\alpha \qquad \ldots(2)$$

Locus of point of intersection of PS and QR can be obtained by eliminating the variable P_2 from (1) and (2)

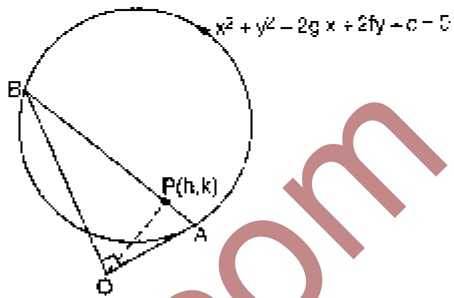
i.e., $$\left(\frac{x}{p_1} - \sec\alpha\right)\frac{x}{y} + \frac{y}{p_1} = \operatorname{cosec}\alpha$$

$$\Rightarrow \quad (x - p_1\sec\alpha)x + y^2 = p_1 y\operatorname{cosec}\alpha$$

$$\Rightarrow \quad x^2 + y^2 - p_1 x\sec\alpha - p_1 y\operatorname{cosec}\alpha = 0$$

which is a circle through origin.

7. Let, $P(h, k)$ be the foot of perpendicular drawn from origin $O(0, 0)$ on the chord AB of the given circle such that the chord AB subtends a right angle at the origin.
The equation of chord AB is,



$$y-k = -\frac{h}{k}(x-h)$$

$$\Rightarrow \quad hx + ky = h^2 + k^2.$$

The combined equation of OA and OB is homogeneous equation of second degree obtained by the help of the given circle and the chord AB and is given by,

$$x^2 + y^2 + (2gx+2fy)\left(\frac{hx+ky}{h^2+k^2}\right) + c\left(\frac{hx+ky}{h^2+k^2}\right)^2 = 0$$

Lines OA and OB given by the above equation are at right angle. Therefore,
coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \quad \left\{1 + \frac{2gh}{h^2+k^2} + \frac{ch^2}{(h^2+k^2)^2}\right\} + \left\{1 + \frac{2fk}{h^2+k^2} + \frac{ck^2}{(h^2+k^2)^2}\right\} = 0$$

$$\Rightarrow \quad 2(h^2+k^2) + 2(gh+fk) + c = 0$$

or $$h^2+k^2+gh+fk+\frac{c}{2} = 0$$

$\therefore$ Required equation of locus is,

$$x^2+y^2+gx+fy+\frac{c}{2}=0$$

8. Let the points $\left(m_i, \frac{1}{m_i}\right)$; $i = 1, 2, 3, 4$ lie on the circle $x^2+y^2+2gx+2fy+c=0$.

Then, $m_i^2 + \frac{1}{m_i^2} + 2gm_i + \frac{2f}{m_i} + c = 0$; $i = 1, 2, 3, 4$

$\Rightarrow m_i^4 + 2gm_i^3 + cm_i^2 + 2fm_i + 1 = 0$; $i = 1, 2, 3, 4$

$\Rightarrow m_1, m_2, m_3$ and m_4 are the roots of the equation

$$m^4 + 2gm^3 + cm^2 + 2fm + 1 = 0$$

$$\Rightarrow \quad m_1m_2m_3m_4 = \frac{1}{1} = 1$$

9. The parametric form of OP is,

$$\frac{x-0}{\cos 45^\circ} = \frac{y-0}{\sin 45^\circ}$$

Since, $OP = 4\sqrt{2}$

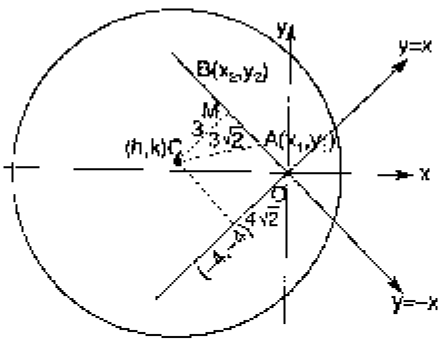
So, the coordinates of P are given by

$$\frac{x-0}{\cos 45^\circ} = \frac{y-0}{\sin 45^\circ} = -4\sqrt{2}$$

So, $P(-4, -4)$

Let $C(h, k)$ be the centre of circle and r be its radius.

Now, $CP \perp OP$



$\Rightarrow \quad \frac{k+4}{h+4}\cdot(1) = -1$

$\Rightarrow \quad k+4 = -h-4$

$\Rightarrow \quad h+k = -8$ ...(1)

Also, $CP^2 = (h+4)^2 + (k+4)^2$

$\Rightarrow \quad (h+4)^2 + (k+4)^2 = r^2$ ...(2)

In ΔACM, we have

$$AC^2 = (3\sqrt{2})^2 + \left(\frac{h+k}{\sqrt{2}}\right)^2$$

$\Rightarrow \quad r^2 = 18 + 32$

$\Rightarrow \quad r = 5\sqrt{2}$ ...(3)

also, $CP = r$

$\Rightarrow \quad \left|\frac{h-k}{\sqrt{2}}\right| = r$

$\Rightarrow \quad h-k = \pm 10$ ...(4)

From (1) and (4), we get

$(h = -9, k = 1)$ or $(h = 1, k = -9)$.

Thus, the equation of the circles are

$(x+9)^2 + (y-1)^2 = (5\sqrt{2})^2$

and $(x-1)^2 + (y+9)^2 = (5\sqrt{2})^2$.

or $x^2 + y^2 + 18x - 2y + 32 = 0$

and $x^2 + y^2 - 2x + 18y + 32 = 0$

Clearly, $(-10, 2)$ lies interior of

$$x^2 + y^2 + 18x - 2y + 32 = 0$$

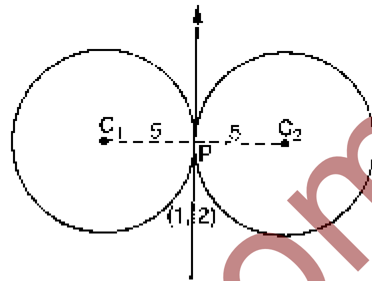
Hence, the required equation of circle is,

$$x^2 + y^2 + 18x - 2y + 32 = 0$$

10. We have,

slope of the common tangent $= -\frac{4}{3}$

$\therefore$ slope of $C_1C_2 = \frac{3}{4}$.



If C_1C_2 makes an angle θ with x-axis, then $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$.

So, the equation of C_1C_2 in parametric form is

$$\frac{x-1}{4/5} = \frac{y-2}{3/5} \quad \text{...(1)}$$

Since, C_1 and C_2 are points on (1) at a distance of 5 units from P.

So, the coordinates of C_1 and C_2 are given by

$$\frac{x-1}{4/5} = \frac{y-2}{3/5} = \pm 5$$

$\Rightarrow \quad x = 1 \pm 4$ and $y = 2 \pm 3$.

Thus, the co-ordinates of C_1 and C_2 are $(5, 5)$ and $(-3, -1)$ respectively.

Hence, the equations of the two circles are

$(x-5)^2 + (y-5)^2 = 5^2$

and $(x+3)^2 + (y+1)^2 = 5^2$

11. Suppose the circles have centres at C_1, C_2 and C_3 with radius R_1, R_2 and R_3 respectively. Let the circles touch at A, B and C. Let the common tangents at A, B and C meet at O. We have $OA = OB = OC = 4$ (given). Now the circle with centre at O and passing through A, B and C is the incircle of the triangle $C_1 C_2 C_3$. (because $OA \perp C_1C_2$)

Therefore, the inradius of $\Delta C_1 C_2 C_3$ is 4.

and $r = \frac{\Delta}{s}$

$\Rightarrow \quad 2s = R_1 + R_2 + R_2 + R_3 + R_3 + R_1$

$\Rightarrow \quad 2s = 2(R_1 + R_2 + R_3)$

$\Rightarrow \quad s = R_1 + R_2 + R_3$

and $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{(R_1 + R_2 + R_3)(R_3)(R_2)(R_1)}$

$\Rightarrow$ but $\quad 4 = \frac{\sqrt{R_1 R_2 R_3 (R_1 + R_2 + R_3)}}{R_1 + R_2 + R_3}$

⇒ $16=\dfrac{R_1 R_2 R_3 (R_1+R_2+R_3)}{(R_1+R_2+R_3)^2}$

⇒ $16=\dfrac{R_1 R_2 R_3}{R_1+R_2+R_3}$

12. The given circle is

$$2x(x-a)+y(2y-b)=0$$

or $$x(x-a)+y(y-b/2)=0$$

or $$x^2+y^2-ax-by/2=0 \quad \ldots(1)$$

Let one of the chord through $(a, b/2)$ be bisected at $(h, 0)$.

Then the equation of the chord having $(h, 0)$ as mid point is

$$T=S_1$$

⇒ $h\cdot x+0\cdot y-\dfrac{a}{2}(x+h)-\dfrac{b}{4}(y+0)=h^2+0-ah-0$

⇒ $$\left(h-\frac{a}{2}\right)x-\frac{by}{4}-\frac{a}{2}h=h^2-ah \quad \ldots(2)$$

Now, (2) will pass through $(a, b/2)$ if

$$(h-a/2)a-\frac{b}{4}\cdot\frac{b}{2}-\frac{a}{2}h=h^2-ah$$

⇒ $$h^2-\frac{3}{2}ah+\frac{a^2}{2}+\frac{b^2}{8}=0 \quad \ldots(3)$$

According to the given condition, (3) must have two distinct real roots. This is possible if the discriminant of (3) is greater than 0. That is, if

$$\frac{9}{4}a^2-4\left(\frac{a^2}{2}+\frac{b^2}{8}\right)>0$$

⇒ $$\frac{a^2}{4}-\frac{b^2}{2}>0$$

⇒ $$a^2>2b^2$$

13. The equation of the circle on the line joining the points $A(3, 7)$ and $B(6, 5)$ as diameter is

$$(x-3)(x-6)+(y-7)(y-5)=0 \quad \ldots(1)$$

and the equation of the line joining the points $A(3, 7)$ and $B(6, 5)$ is

$$y-7=\frac{7-5}{3-6}(x-3)$$

⇒ $$2x+3y-27=0 \quad \ldots(2)$$

Now the equation of family of circles passing through the point of intersection of (1) and (2) is

$$S+\lambda P=0$$

⇒ $(x-3)(x-6)+(y-7)(y-5)+\lambda(2x+3y-27)=0$

⇒ $x^2-6x-3x+18+y^2-5y-7y+35+2\lambda x+3\lambda y-27\lambda=0$

⇒ $S_1\equiv x^2+y^2+x(2\lambda-9)+y(3\lambda-12)+(53-27\lambda)=0 \quad \ldots(3)$

Again the circle which cuts the members of family of circles is

$$S_2\equiv x^2+y^2-4x-6y-3=0 \quad \ldots(4)$$

and the equation of common chord to circles S_1 and S_2 is

$$S_1-S_2=0$$

⇒ $\{x(2\lambda-9)+y(3\lambda-12)+(53-27\lambda)\}-\{-4x-6y-3\}=0$

⇒ $x(2\lambda-9+4)+y(3\lambda-12+6)+(53-27\lambda+3)=0$

⇒ $2\lambda x-5x+3\lambda y-6y+56-27\lambda=0$

⇒ $(-5x-6y+56)+\lambda(2x+3y-27)=0$

which represents equations of two straight lines passing through the fixed point whose coordinates are obtained by solving the two equations

$$5x+6y-56=0 \text{ and } 2x+3y-27=0$$

Solving for x and y, we get $x=2$ and $y=23/3$.

14. Two circles touch each other externally if $C_1C_2=r_1+r_2$ and internally if $C_1C_2=r_1-r_2$

and $x^2+y^2-4x-2y=-4$ (given)

⇒ $C_1(2, 1)$ and $r_1=1$

and $x^2+y^2-12x-8y=-36$ (given)

⇒ $C_2(6, 4)$ and $r_2=4$

The distance between the centres is

$$\sqrt{(6-2)^2+(4-1)^2}=\sqrt{16+9}=5$$

⇒ $$C_1C_2=r_1-r_2$$

Therefore, the circles touch each other externally and at the point of touching the point divides the line joining the two centres internally in the ratio of their radii, 1 : 4.

Therefore, $x_1=\dfrac{1\times6+4\times2}{1+4}=\dfrac{14}{5}$

$y_1=\dfrac{1\times4+4\times1}{1+4}=\dfrac{8}{5}$

Again to determine the equations of common tangents touching the circles in distinct points, we know that the tangents pass through a point which divides the line joining the two centres externally in the ratio of their radii *i.e.* 1 : 4.

Hence, $x_2=\dfrac{1\times6-4\times2}{1-4}=\dfrac{-2}{-3}=\dfrac{2}{3}$

and $y_2=\dfrac{1\times4-4\times1}{1-4}=0$

Now, let m be the slope of the tangent. This line passing through $(2/3, 0)$ is

⇒ $$y-0=m(x-2/3)$$

⇒ $$y-mx+\frac{2}{3}m=0$$

but for any tangent, $p=r$ is satisfied.

Therefore, $\dfrac{1-2m+(2/3)m}{\sqrt{1+m^2}}=1$ $[C_1\equiv(2, 1) \text{ and } r_1=1]$

$\Rightarrow \quad 1 - 2m + (2/3)$ [illegible]

$$\Rightarrow \quad 1 - \frac{4}{3}m = \sqrt{1+m^2}$$

$$\Rightarrow \quad 1 + \frac{16}{9}m^2 - \frac{8}{3}m = 1 + m^2$$

$$\Rightarrow \quad \frac{7}{9}m^2 - \frac{8}{3}m = 0$$

$$\Rightarrow \quad m\left(\frac{7}{9}m - \frac{8}{3}\right) = 0$$

$$\Rightarrow \quad m = 0,\ m = \frac{24}{7}$$

Hence, the equations of the two tangents are

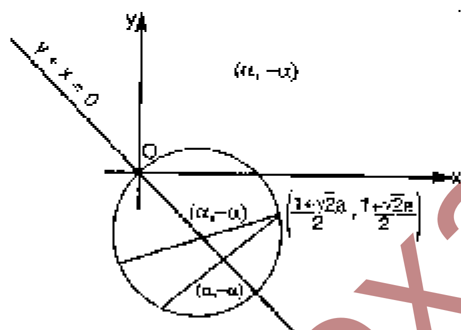
$$y = 0 \quad \text{and} \quad y = \frac{24}{7}\left(x - \frac{2}{3}\right)$$

$$\Rightarrow \quad y = 0 \quad \text{and} \quad 7y - 24x + 16 = 0$$

15. $2x^2 + 2y^2 - (1+\sqrt{2}a)x - (1-\sqrt{2}a)y = 0$

$$\Rightarrow \quad x^2 + y^2 - \left(\frac{1+\sqrt{2}a}{2}\right)x - \left(\frac{1-\sqrt{2}a}{2}\right)y = 0$$

Since, $y + x = 0$ bisects two chords of this circle, mid-points of the chords must be of the form $(\alpha, -\alpha)$.



Equation of the chord having $(\alpha, -\alpha)$ as mid-points is

$$T = S_1$$

$$\Rightarrow \quad x\alpha + y(-\alpha) - \left(\frac{1+\sqrt{2}a}{4}\right)(x+\alpha) - \left(\frac{1-\sqrt{2}a}{4}\right)(y-\alpha)$$

$$= \alpha^2 + (-\alpha)^2 - \left(\frac{1+\sqrt{2}a}{2}\right)\alpha - \left(\frac{1-\sqrt{2}a}{2}\right)(-\alpha)$$

$$\Rightarrow \quad 4x\alpha - 4y\alpha - (1+\sqrt{2}a)x - (1+\sqrt{2}a)\alpha - (1-\sqrt{2}a)y + (1-\sqrt{2}a)\alpha$$

$$= 4\alpha^2 + 4\alpha^2 - (1+\sqrt{2}a)\cdot 2\alpha + (1-\sqrt{2}a)\cdot 2\alpha$$

$$\Rightarrow \quad 4\alpha x - 4\alpha y - (1+\sqrt{2}a)x - (1-\sqrt{2}a)y$$

$$= 8\alpha^2 - (1+\sqrt{2}a)\alpha + (1-\sqrt{2}a)\alpha$$

But this chord will pass through the point

$$\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$$

$$4\alpha\left(\frac{1+\sqrt{2}a}{2}\right) - 4\alpha\left(\frac{1-\sqrt{2}a}{2}\right) - \frac{(1+\sqrt{2}a)(1+\sqrt{2}a)}{2} - \frac{(1-\sqrt{2}a)(1-\sqrt{2}a)}{2}$$

$$= 8\alpha^2 - 2\sqrt{2}a\alpha$$

$$\Rightarrow \quad 2\alpha\,[(1+\sqrt{2}a - 1 + \sqrt{2}a)] = 8\alpha^2 - 2\sqrt{2}a\alpha$$

$$\Rightarrow \quad 4\sqrt{2}a\alpha - \frac{1}{2}[2 + 2(\sqrt{2}a)^2] = 8\alpha^2 - 2\sqrt{2}a\alpha$$

$$[\because (a+b)^2 + (a-b)^2 = 2a^2 + 2b^2]$$

$$\Rightarrow \quad 8\alpha^2 - 6\sqrt{2}a\alpha + 1 + 2a^2 = 0$$

But this quadratic equation will have two distinct roots if

$$(6\sqrt{2}a)^2 - 4(8)(1+2a^2) > 0$$

$$\Rightarrow \quad 72a^2 - 32(1+2a^2) > 0$$

$$\Rightarrow \quad 72a^2 - 32 - 64a^2 > 0$$

$$\Rightarrow \quad 8a^2 - 32 > 0$$

$$\Rightarrow \quad a^2 - 4 > 0$$

$$\Rightarrow \quad a^2 > 4 \quad \Rightarrow \quad a < -2 \cup a > 2$$

Therefore, $a \in (-\infty, -2) \cup (2, \infty)$.

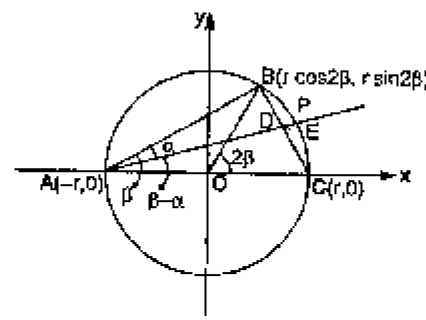
16. Let the radius of the circle be r. Take x-axis along AC and the $O(0, 0)$ as centre of the circle. Therefore, coordinate of A and C are $(-r, 0)$ and $(r, 0)$ respectively

Now, $\angle BAC = \beta$, $\angle BOC = 2\beta$

Therefore, coordinates of B are $(r\cos 2\beta, r\sin 2\beta)$.

And slope of AD is $\tan(\beta - \alpha)$.

Let (x, y) be the coordinates of the point D. Equation of AD is



$$y = \tan(\beta - \alpha)(x + r) \qquad \ldots(1)$$

($\because$ slope $= \tan(\beta - \alpha)$ and point is $(-r, 0)$)

Now, Equation of BC is

$$y = \frac{r\sin 2\beta - 0}{r\cos 2\beta - r}(x - r)$$

$$\Rightarrow \quad y = \frac{r\cdot 2\sin\beta\cos\beta}{r(-2\sin^2\beta)}(x - r)$$

$$\Rightarrow \quad y = \frac{2\sin\beta\cos\beta}{-2\sin^2\beta}(x - r)$$

$$\Rightarrow \quad y = -\cot\beta\,(x - r) \qquad \ldots(2)$$

To obtain the coordinate of D, solve (1) and (2) simultaneously

$$\Rightarrow \quad \tan(\beta-\alpha)(x+r) = -\cot\beta\,(x-r)$$

$$\Rightarrow \quad x\tan(\beta-\alpha) + r\tan(\beta-\alpha) = -x\cot\beta + r\cot\beta$$

$$\Rightarrow \quad x[\tan(\beta-\alpha)+\cot\beta] = r[\cot\beta - \tan(\beta-\alpha)]$$

$$\Rightarrow \quad x\left[\frac{\sin(\beta-\alpha)}{\cos(\beta-\alpha)} + \frac{\cos\beta}{\sin\beta}\right] = r\left[\frac{\cos\beta}{\sin\beta} - \frac{\sin(\beta-\alpha)}{\cos(\beta-\alpha)}\right]$$

$$\Rightarrow \quad x\left[\frac{\sin(\beta-\alpha)\sin\beta + \cos(\beta-\alpha)\cos\beta}{\cos(\beta-\alpha)\sin\beta}\right] = r\left[\frac{\cos\beta\cos(\beta-\alpha) - \sin\beta\sin(\beta-\alpha)}{\sin\beta\cos(\beta-\alpha)}\right]$$

$$\Rightarrow \quad x[\cos(\beta-\alpha)\cos\beta + \sin(\beta-\alpha)\sin\beta] = r[\cos(\beta-\alpha)\cos\beta - \sin\beta\sin(\beta-\alpha)]$$

$$\Rightarrow \quad x[\cos(\beta-\alpha-\beta)] = r[\cos(\beta-\alpha+\beta)]$$

$$\Rightarrow \quad x = \frac{r\cos(2\beta-\alpha)}{\cos\alpha}$$

Putting this value in (2), we get

$$y = -\cot\beta\left[\frac{r\cos(2\beta-\alpha)}{\cos\alpha} - r\right]$$

$$\Rightarrow \quad y = -\frac{\cos\beta\cdot r}{\sin\beta}\left[\frac{\cos(2\beta-\alpha)-\cos\alpha}{\cos\alpha}\right]$$

$$\Rightarrow y = -\frac{r\cos\beta}{\sin\beta}\left[\frac{2\sin\frac{2\beta-\alpha+\alpha}{2}\sin\frac{\alpha-2\beta+\alpha}{2}}{\cos\alpha}\right]$$

$$\Rightarrow y = -\frac{r\cos\beta}{\sin\beta}\left[\frac{2\sin\beta\cdot\sin(\alpha-\beta)}{\cos\alpha}\right]$$

$$= -2r\cos\beta\sin(\alpha-\beta)/\cos\alpha$$

Therefore, coordinates of D are

$$\left(\frac{r\cos(2\beta-\alpha)}{\cos\alpha}, -\frac{2r\cos\beta\sin(\alpha-\beta)}{\cos\alpha}\right)$$

Thus, coordinates of E are

$$\left(\frac{r\cos(2\beta-\alpha)+r\cos\alpha}{2\cos\alpha}, -r\frac{\cos\beta\sin(\alpha-\beta)}{\cos\alpha}\right)$$

$$\Rightarrow \quad \left(r\frac{2\cos\left(\frac{2\beta-\alpha+\alpha}{2}\right)\cos\left(\frac{2\beta-\alpha-\alpha}{2}\right)}{2\cos\alpha}, \; r\frac{\cos\beta\sin(\beta-\alpha)}{\cos\alpha}\right)$$

$$\Rightarrow \quad \left(r\cdot\frac{\cos\beta\cos(\beta-\alpha)}{\cos\alpha}, \; r\frac{\cos\beta\sin(\beta-\alpha)}{\cos\alpha}\right)$$

Since, $AE = d$, we get

$$d^2 = r^2\left[\frac{\cos\beta\cos(\beta-\alpha)}{\cos\alpha}+1\right]^2 + r^2\left[\frac{\cos\beta\sin(\beta-\alpha)}{\cos\alpha}\right]^2$$

$$= \frac{r^2}{\cos^2\alpha}[\cos^2\beta\cos^2(\beta-\alpha) + \cos^2\alpha + 2\cos\beta\cos(\beta-\alpha)\cos\alpha + \cos^2\beta\sin^2(\beta-\alpha)]$$

$$= \frac{r^2}{\cos^2\alpha}[\cos^2\beta\{\cos^2(\beta-\alpha)+\sin^2(\beta-\alpha)\} + \cos^2\alpha + 2\cos\beta\cos\alpha\cos(\beta-\alpha)]$$

$$= \frac{r^2}{\cos^2\alpha}[\cos^2\beta + \cos^2\alpha + 2\cos\alpha\cos\beta\cos(\beta-\alpha)]$$

$$\Rightarrow \quad r^2 = \frac{d^2\cos^2\alpha}{\cos^2\beta+\cos^2\alpha+2\cos\alpha\cos\beta\cos(\beta-\alpha)}$$

Therefore, area of the circle

$$= \pi r^2$$

$$= \frac{\pi d^2\cos^2\alpha}{\cos^2\beta+\cos^2\alpha+2\cos\alpha\cos\beta\cos(\beta-\alpha)}$$

17. The given circle is,

$$ax^2 + 2hxy + by^2 = 1 \qquad \ldots(1)$$

Let the point P not lying on (1) be (x_1, y_1), let θ be the inclination of line through P which intersects the given curve at Q and R. Then equation of line through P is,

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$$

$$\Rightarrow \quad x = x_1 + r\cos\theta, \; y = y_1 + r\sin\theta$$

for point Q and R, above point must lie on (1)

$$\Rightarrow \quad a(x_1+r\cos\theta)^2 + 2h(x_1+r\cos\theta)(y_1+r\sin\theta) + b(y_1+r\sin\theta)^2 = 1$$

$$\Rightarrow \quad (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)r^2 + 2(ax_1\cos\theta + hx_1\sin\theta + hy_1\cos\theta + by_1\sin\theta)r + (ax_1^2 + 2hx_1y_1 + by_1^2 - 1) = 0$$

It is quadratic in r, giving two values of r as PQ and PR.

$$\therefore \quad PQ\cdot PR = \frac{ax_1^2+2hx_1y_1+by_1^2-1}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta}$$

Here, $ax_1^2 + 2hx_1y_1 + by_1^2 - 1 \neq 0$, as (x_1, y_1) does not lie on (1)

Also, $a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$

$$= a + 2h\sin\theta\cos\theta + (b-a)\sin^2\theta$$

$$= a + \sin\theta\,(2h\cos\theta + (b-a)\sin\theta)$$

$$= a + \sin\theta\cdot\sqrt{4h^2+(b-a)^2}\cdot\{\cos\theta\sin\phi + \sin\theta\cos\phi\}$$

where, $\tan\theta = \dfrac{b-a}{2h}$

$$= a + \sqrt{4h^2+(b-a)^2}\,\sin\theta\sin(\theta+\phi)$$

which will be independent of θ, if

$$4h^2 + (b-a)^2 = 0$$

$$\Rightarrow \quad h = 0 \quad \text{and} \quad b = a$$

$\therefore$ Equation (1) reduces to

$$x^2 + y^2 = \frac{1}{a}$$

which is a circle.

18. Equations of any circle C with centre at $(0, \sqrt{2})$ is given by

$$(x-0)^2 + (y-\sqrt{2})^2 = r^2$$

or $$x^2 + y^2 - 2\sqrt{2}\, y + 2 = r^2 \qquad ...(1)$$

where $r > 0$

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be three distinct rational points on (1). Since a straight line parallel to x-axis meets a circle in at most two points, either y_1, y_2 or y_1, y_3 Putting these in (1), we get

$$x_1^2 + y_1^2 - 2\sqrt{2}\, y_1 = r^2 - 2 \qquad ...(2)$$

$$x_2^2 + y_2^2 - 2\sqrt{2}\, y_2 = r^2 - 2 \qquad ...(3)$$

$$x_3^2 + y_3^2 - 2\sqrt{2}\, y_3 = r^2 - 2 \qquad ...(4)$$

Subtracting (2) from (3), we obtain

$$p_1 - \sqrt{2}\, q_1 = 0$$

where $$p_1 = x_2^2 + y_2^2 - x_1^2 - y_1^2,$$

$$q_1 = y_2 - y_1$$

Subtracting (2) from (4), we obtain

$$p_2 - \sqrt{2} q_2 = 0$$

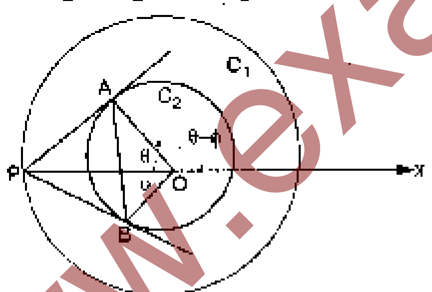
where $p_2 = x_3^2 + y_3^2 - x_1^2 - y_1^2$, $q_2 = y_3 - y_1$

Now, p_1, p_2, q_1, q_2 are rational numbers. Also either $q_1 \neq 0$ or $q_2 \neq 0$. If $q_1 \neq 0$ then $\sqrt{2} = p_1 / q_1$ and if $q_2 \neq 0$, then $\sqrt{2} = p_2 / q_2$. In any case $\sqrt{2}$ is a rational number. This is a contradiction.

19. Let the point P be $(2r\cos\theta, 2r\sin\theta)$

We have $OA = r$, $OP = 2r$

As $\Delta\, OAP$ is a right angled triangle,



$$\cos\phi = 1/2 \Rightarrow \phi = \pi/3$$

$\therefore$ Coordinates of A are $\{r\cos(\theta - \pi/3), r\sin(\theta - \pi/3)\}$ and that of B are $[r\cos(\theta + \pi/3), r\sin(\theta + \pi/3)]$

If p, q is the centroid of $\Delta\, PAB$,

then $$p = \frac{1}{3}[r\cos(\theta - \pi/3) + r\cos(\theta + \pi/3) + 2r\cos\theta]$$

$$= \frac{1}{3}[r\{\cos(\theta - \pi/3) + \cos(\theta + \pi/3)\} + 2r\cos\theta]$$

$$= \frac{1}{3}\left[r\left(2\cos\frac{\theta - \frac{\pi}{3} + \theta + \frac{\pi}{3}}{2} \cdot \cos\frac{\theta - \frac{\pi}{3} - \theta - \frac{\pi}{3}}{2}\right) + 2r\cos\theta\right]$$

$$= \frac{1}{3}[r\{2\cos\theta\cos\pi/3\} + 2r\cos\theta]$$

$$= \frac{1}{3}[r \cdot \cos\theta + 2r\cos\theta] = r\cos\theta$$

and $$q = \frac{1}{3}\left[r\sin\left(\theta - \frac{\pi}{3}\right) + r\sin\left(\theta + \frac{\pi}{3}\right) + 2r\sin\theta\right]$$

$$= \frac{1}{3}\left[r\left\{\sin\left(\theta - \frac{\pi}{3}\right) + \sin\left(\theta + \frac{\pi}{3}\right)\right\} + 2r\sin\theta\right]$$

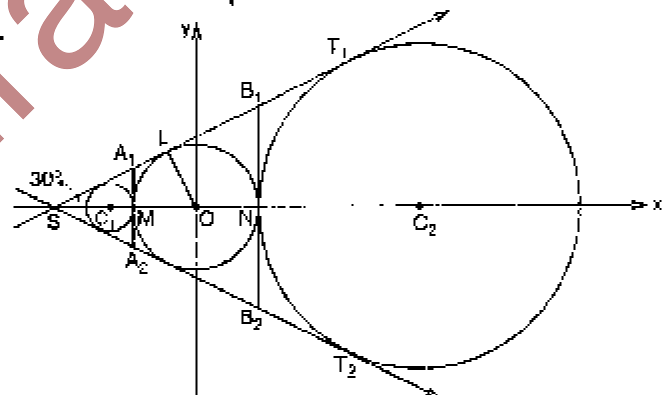
$$= \frac{1}{3}\left[r\left(2\sin\frac{\theta - \frac{\pi}{3} + \theta + \frac{\pi}{3}}{2} \cdot \cos\frac{\theta - \frac{\pi}{3} - \theta - \frac{\pi}{3}}{2}\right) + 2r\sin\theta\right]$$

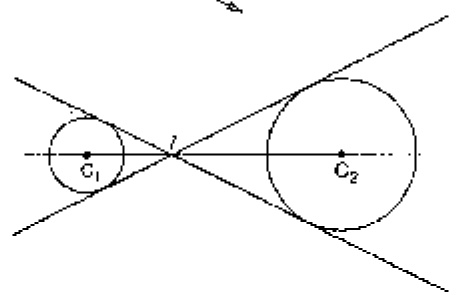
$$= \frac{1}{3}[r\{2\sin\theta\cos\pi/3\} + 2r\sin\theta]$$

$$= \frac{1}{3}[r\{\sin\theta\} + 2r\sin\theta] = r\sin\theta$$

Now, $(p, q) = (r\cos\theta, r\sin\theta)$ lies on $x^2 + y^2 = r^2$ which is called C_1.

20.





From figure it is clear that triangle OLS is a right triangle with right angle at L.

Also $OL = 1$ and $OS = 2$

$$\therefore \quad \sin(\angle LSO) = \frac{1}{2} \Rightarrow \angle LSO = 30°$$

Since $SA_1 = SA_2$, $\Delta\, SA_1A_2$ is an equilateral triangle. The circle with centre at C_1 is a circle inscribed in the $\Delta\, SA_1A_2$. Therefore, centre C_1 is centroid of $\Delta\, SA_1A_2$. This, C_1 divides SM in the ratio 2 : 1. Therefore, coordinates of C_1 are $(-4/3, 0)$ and its radius $= C_1M = 1/3$

∴ its equation is $(x+4/3)^2 + y^2 = (1/3)^2$ ...(1)

The other circle touches the equilateral triangle SB_1B_2 externally. Its radius r is given by $r = \frac{\Delta}{s-a}$, where $B_1B_2 = a$. But $\Delta = \frac{1}{2}(a)(SN) = \frac{3}{2}a$

and $s - a = \frac{3}{2}a - a = a/2$

Thus, $r = 3$

⇒ Coordinates of C_2 are (4, 0)

∴ equation of circle with centre at C_2 is

$$(x-4)^2 + y^2 = 3^2 \quad ...(2)$$

Equations of common tangents to circle (1) and circle C are

$x = -1$ and $y = \pm\frac{1}{\sqrt{3}}(x+2)$ [T_1 and T_2]

Equation of common tangents to circle (2) and circle C are

$x = 1$ and $y = \pm\frac{1}{\sqrt{3}}(x+2)$ [T_1 and T_2]

Two tangents common to (1) and (2) are T_1 and T_2 at O.
To find the remaining two transverse tangents to (1) and (2), we find a point I which divides the joint of C_1C_2 in the ratio $r_1 : r_2 = 1/3 : 3 = 1 : 9$
Therefore, coordinates of I are $(-4/5, 0)$

Equation of any line through I is $y = m(x + 4/5)$. It will touch (1) if

$$\frac{\left|m\left(\frac{-4}{3}+\frac{4}{5}\right)-0\right|}{\sqrt{1+m^2}} = \frac{1}{3}$$

⇒ $\left|-\frac{8m}{15}\right| = \frac{1}{3}\sqrt{1+m^2}$

⇒ $64m^2 = 25(1+m^2)$

⇒ $39m^2 = 25$

⇒ $m = \pm\frac{5}{\sqrt{39}}$

Therefore, these tangents are

$$y = \pm\frac{5}{\sqrt{39}}\left(x+\frac{4}{5}\right)$$

21. Equation of any tangent to circle $x^2 + y^2 = r^2$ is

$$x\cos\theta + y\sin\theta = r \quad ...(1)$$

Suppose (1) is tangent to $4x^2 + 25y^2 = 100$

or $\frac{x^2}{25} + \frac{y^2}{4} = 1$ at (x_1, y_1)

Then (1) and $\frac{xx_1}{25} + \frac{yy_1}{4} = 1$ are identical

∴ $\frac{x_1/25}{\cos\theta} = \frac{y_1/4}{\sin\theta} = \frac{1}{r}$

⇒ $x_1 = \frac{25\cos\theta}{r}$, $y_1 = \frac{4\sin\theta}{r}$

The line (1) meet the coordinates axes in $A(r\sec\theta, 0)$ and $B(0, r\,\text{cosec}\,\theta)$. Let (h, k) be mid point of AB.

Then $h = \frac{r\sec\theta}{2}$ and $k = \frac{r\,\text{cosec}\,\theta}{2}$

Therefore, $2h = \frac{r}{\cos\theta}$ and $2k = \frac{r}{\sin\theta}$

∴ $x_1 = \frac{25}{2h}$ and $y_1 = \frac{4}{2k}$

As (x_1, y_1) lies on the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$

we get $\frac{1}{25}\left(\frac{625}{4h^2}\right) + \frac{1}{4}\left(\frac{4}{k^2}\right) = 1$

⇒ $\frac{25}{4h^2} + \frac{1}{k^2} = 1$

or $25k^2 + 4h^2 = 4h^2k^2$

Therefore, required locus is $4x^2 + 25y^2 = 4x^2y^2$

22. $2x^2 + y^2 - 3xy = 0$ (given)

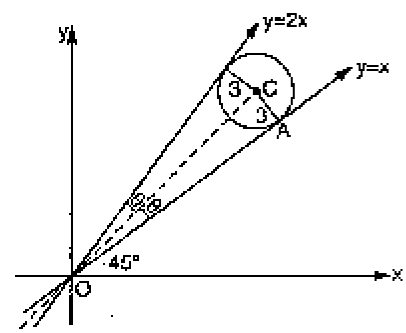
⇒ $2x^2 - 2xy - xy + y^2 = 0$

⇒ $2x(x-y) - y(x-y) = 0$

⇒ $(2x-y)(x-y) = 0$

⇒ $y = 2x$, $y = x$ are the equations of straight lines passing through origin.

Now, let the angle between the lines be 2θ and the line $y = x$ makes angle of 45° with x-axis
Therefore $\tan(45° + 2\theta) = 2$ (slope of the line $y = 2x$)



⇒ $\frac{\tan 45° - \tan 2\theta}{1 - \tan 45° \times \tan 2\theta} = 2$

⇒ $\frac{1 + \tan 2\theta}{1 - \tan 2\theta} = 2$

⇒ $\frac{(1+\tan 2\theta) - (1 - \tan 2\theta)}{(1+\tan 2\theta) + (1 - \tan 2\theta)} = \frac{2-1}{(2+1)} = \frac{1}{3}$

⇒ $\frac{2\tan 2\theta}{2} = \frac{1}{3}$ ⇒ $\tan 2\theta = \frac{1}{3}$

⇒ $\frac{2\tan\theta}{1-\tan^2\theta} = \frac{1}{3}$ ⇒ $(2\tan\theta)\cdot 3 = 1 - \tan^2\theta$

⇒ $\tan^2\theta + 6\tan\theta - 1 = 0$

$\Rightarrow \quad \tan\theta = \dfrac{-6 \pm \sqrt{36 + 4\times1\times1}}{2}$

$\Rightarrow \quad \tan\theta = \dfrac{-6 \pm \sqrt{40}}{2}$

$\Rightarrow \quad \tan\theta = -3 \pm \sqrt{10}$

$\Rightarrow \quad \tan\theta = -3 + \sqrt{10} \qquad (\because 0 < \theta < \pi/4)$

Again in $\Delta\, OCA$

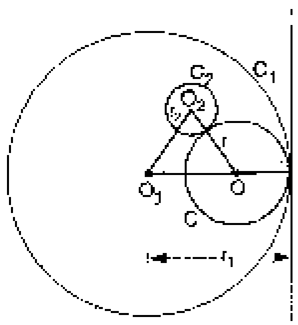
$\tan\theta = \dfrac{3}{OA}$; $OA = \dfrac{3}{\tan\theta} = \dfrac{3}{(-3+\sqrt{10})}$

$\therefore \quad = \dfrac{3\,(3+\sqrt{10})}{(-3+\sqrt{10})\,(3+\sqrt{10})}$

$= \dfrac{3\,(3+\sqrt{10})}{(10-9)} = 3\,(3+\sqrt{10})$

23. Let the given circles C_1 and C_2 have centres O_1 and O_2 and radii r_1 and r_2 respectively.

Let the variable circle C touching C_1 internally, C_2 externally have a radius r and centre at O.



Now, $\quad OO_2 = r + r_2$ and $OO_1 = r_1 - r$

$\Rightarrow \quad OO_1 + OO_2 = r_1 + r_2$

which is greater than O_1O_2 as $O_1O_2 < r_1 + r_2$

$(\because C_2$ lies inside $C_1)$

$\Rightarrow$ locus of O is an ellipse with foci O_1 and O_2.

Alternate solution :

Let equations of C_1 be $x^2 + y^2 = r_1^2$ and of C_2 be $(x-a)^2 + (y-b)^2 = r_2^2$

Let centre C be (h, k) and radius r, then by the given condition

$\sqrt{(h-a)^2 + (k-b)^2} = r + r_2$ and $\sqrt{h^2 + k^2} = r_1 - r$

$\Rightarrow \quad \sqrt{(h-a)^2 + (k-b)^2} + \sqrt{h^2 + k^2} = r_1 + r_2$

Required locus is

$\sqrt{(x-a)^2 + (y-b)^2} + \sqrt{x^2 + y^2} = r_1 + r_2$

which represents an ellipse whose foci are at (a, b) and $(0, 0)$.

24. The equation of circle having tangent $2x + 3y + 1 = 0$ at $(1, -1)$

$\Rightarrow \quad (x-1)^2 + (y+1)^2 + \lambda(2x + 3y + 1) = 0$

$x^2 + y^2 + 2x(\lambda - 1) + y(3\lambda + 2) + (\lambda + 2) = 0 \qquad \ldots(1)$

which is orthogonal to the circle having end point of diameter $(0, -1)$ and $(-2, 3)$.

$\Rightarrow \quad x(x+2) + (y+1)(y-3) = 0$

or $\quad x^2 + y^2 + 2x - 2y - 3 = 0 \qquad \ldots(2)$

$\therefore \quad \dfrac{2(2\lambda - 2)}{2}\cdot 1 + \dfrac{2(3\lambda + 2)}{2}(-1) = \lambda + 2 - 3$

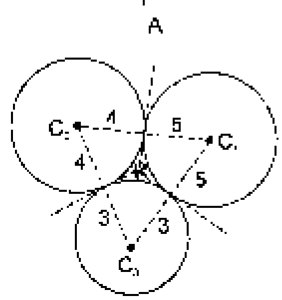
$\Rightarrow \quad 2\lambda - 2 - 3\lambda - 2 = \lambda - 1$

$\Rightarrow \quad 2\lambda = -3 \quad \Rightarrow \quad \lambda = -3/2$

$\therefore$ from equation (1) equation of circle.

$2x^2 + 2y^2 - 10x - 5y + 1 = 0$

25. As the circles with radii 3, 4 and 5 touch each other externally and P is the point of intersection of tangents



$\Rightarrow \quad P$ is incentre of $\Delta C_1C_2C_3$.

Thus distance of point P from the points of contact = In radius (r) of $\Delta\, C_1C_2C_3$.

i.e., $\quad r = \dfrac{\Delta}{s} = \dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$

where $\quad 2s = 7 + 8 + 9$

$\therefore \quad s = 12$

Hence, $\quad r = \sqrt{\dfrac{(12-7)(12-8)(12-9)}{12}} = \sqrt{\dfrac{5\cdot4\cdot3}{12}} = \sqrt{5}$

G ASSERTION AND REASON

1. Since the tangents are perpendicular.
So, locus of perpendicular tangents to circle

$x^2 + y^2 = 169$ is a director circle having equation

$x^2 + y^2 = 338$.

H MATCH THE COLUMN

1. (A) When two circles are intersecting they have a common normal and common tangent.
(B) Two mutually external circles have a common normal and common tangent.
(C) When one circle lies inside of other then, they have a common normal but no common tangent.
(D) Two branches of a hyperbola have a common normal but no common tangent.