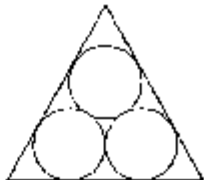


**A** FILL IN THE BLANKS

- The area enclosed within the curve  $|x| + |y| = 1$  is ..... (IIT 1981; 2M)
- The area of the triangle formed by the positive  $x$ -axis and the normal and the tangent to the circle  $x^2 + y^2 = 4$  at  $(1, \sqrt{3})$  is ... (IIT 1989; 2M)

**C** OBJECTIVE QUESTIONS

→ Only one option is correct :

- The area bounded by the curves  $y = f(x)$ , the  $x$ -axis and the ordinates  $x = 1$  and  $x = b$  is  $(b-1) \sin(3b+4)$ . Then  $f(x)$  is : (IIT 1982; 2M)
  - $(x-1) \cos(3x+4)$
  - $\sin(3x-4)$
  - $\sin(3x+4) - 3(x-1) \cos(3x+4)$
  - none of these
- The triangle formed by the tangent to the curve  $f(x) = x^2 + bx - b$  at the point  $(1, 1)$  and the coordinate axes, lies in the first quadrant, If its area is 2, then the value of  $b$  is : (IIT 2001)
  - 1
  - 3
  - 3
  - 1
- The area bounded by the curves  $y = |x| - 1$  and  $y = -|x| + 1$  is : (IIT 2002)
  - 1
  - 2
  - $2\sqrt{2}$
  - 4
- The area of the quadrilateral formed by the tangents at the end points of latus rectum to the ellipse  $\frac{x^2}{9} + \frac{y^2}{5} = 1$ , is : (IIT 2003)
  - $27/4$  sq. units
  - 9 sq. units
  - $27/2$  sq. units
  - 27 sq. units
- The area bounded by the curves  $y = \sqrt{x}$ ,  $2y + 3 = x$  and  $x$ -axis in the 1st quadrant is : (IIT 2003)
  - 9
  - $27/4$
  - 36
  - 18
- The area enclosed between the curves  $y = ax^2$  and  $x = ay^2$  ( $a > 0$ ) is 1 sq. unit. Then the value of 'a' is : (IIT 2004)
  - $\frac{1}{\sqrt{3}}$
  - $\frac{1}{2}$
  - 1
  - $\frac{1}{3}$
- The area of the equilateral triangle, in which three coins of radius 1 cm are placed, as shown in the figure, is : (IIT 2005)
 
  - $6 + 4\sqrt{3}$
  - $4\sqrt{3} - 6$
  - $7 + 4\sqrt{3}$
  - $4\sqrt{3}$
- The area bounded by the curves  $y = (x-1)^2$ ,  $y = (x+1)^2$  and  $y = \frac{1}{4}$  is : (IIT 2005)
  - $\frac{1}{3}$
  - $\frac{2}{3}$
  - $\frac{1}{4}$
  - $\frac{1}{5}$

**D** OBJECTIVE QUESTIONS

→ More than one options are correct :

- For which of the following values of  $m$ , is the area of the region bounded by the curve  $y = x - x^2$  and the line  $y = mx$  equals  $\frac{9}{2}$ ? (IIT 1999; 3M)
  - 4
  - 2
  - 2
  - 4

## E SUBJECTIVE QUESTIONS

- Find the area bounded by the curve  $x^2 = 4y$  and the straight line  $x = 4y - 2$  (IIT 1981; 4M)
- Find the area bounded by the  $x$ -axis, part of the curve  $y = \left(1 + \frac{8}{x^2}\right)$  and the ordinates at  $x = 2$  and  $x = 4$ . If the ordinate at  $x = a$  divides the area into two equal parts, find  $a$ . (IIT 1983; 3M)
- Find the area of the region bounded by the  $x$ -axis and the curves defined by :
 
$$y = \tan x, \quad -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$$

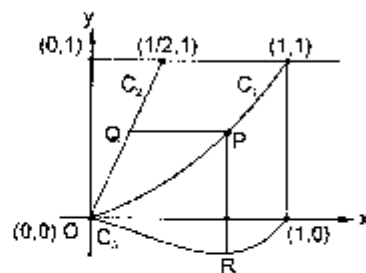
$$y = \cot x, \quad \frac{\pi}{6} \leq x \leq \frac{\pi}{2}$$
 (IIT 1984; 4M)
- Sketch the region bounded by the curves  $y = \sqrt{5 - x^2}$  and  $y = |x - 1|$  and find its area. (IIT 1985; 5M)
- Find the area bounded by the curves  $x^2 + y^2 = 4$ ,  $x^2 = -\sqrt{2}y$  and  $x = y$ . (IIT 1986; 5M)
- Find the area bounded by the curves,  $x^2 + y^2 = 25$ ,  $4y = |4 - x^2|$  and  $x = 0$  above the  $x$ -axis (IIT 1987; 6M)
- Find the area of the region bounded by the curve  $C: y = \tan x$ , tangent drawn to  $C$  at  $x = \frac{\pi}{4}$  and the  $x$ -axis. (IIT 1988; 5M)
- Find all maxima and minima of the function  $y = x(x - 1)^2$ ,  $0 \leq x \leq 2$   
Also determine the area bounded by the curve  $y = x(x - 1)^2$ , the  $y$ -axis and the line  $x = 2$ . (IIT 1989)
- Compute the area of the region bounded by the curves,  $y = ax \ln x$  and  $y = \frac{\ln x}{ex}$  where  $\ln e = 1$ . (IIT 1990; 4M)
- Sketch the curves and identify the region bounded by  $x = \frac{1}{2}$ ,  $x = 2$ ,  $y = \ln x$  and  $y = 2^x$ . Find the area of this region. (IIT 1991; 4M)
- Sketch the region bounded by the curves  $y = x^2$  and  $y = 2/(1 - x^2)$   
Find its area. (IIT 1992; 4M)
- In what ratio does the  $x$ -axis divide the area of the region bounded by the parabolas  $y = 4x - x^2$  and  $y = x^2 - x$ ? (IIT 1994; 5M)
- Consider a square with vertices at  $(1, 1)$ ,  $(-1, 1)$ ,  $(-1, -1)$  and  $(1, -1)$ . Let  $S$  be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region  $S$  and find its area. (IIT 1995; 5M)
- Let  $A_n$  be the area bounded by the curve  $y = (\tan x)^n$  and the lines  $x = 0$ ,  $y = 0$  and  $x = \frac{\pi}{4}$ . Prove that for

$$n > 2, A_n + A_{n+2} = \frac{1}{n+1}$$

$$\text{and deduce } \frac{1}{2n+2} < A_n < \frac{1}{2n-2}$$

(IIT 1996; 3M)

- Find all the possible values of  $b > 0$  so that the area of the bounded region enclosed between the parabolas  $y = x - bx^2$  and  $y = \frac{x^2}{b}$  is maximum. (IIT 1997C; 5M)
- Let  $O(0, 0)$ ,  $A(2, 0)$  and  $B\left(1, \frac{1}{\sqrt{2}}\right)$  be the vertices of a triangle. Let  $R$  be the region consisting of all those points  $P$  inside  $\Delta OAB$  which satisfy  $d(P, OA) \geq \min\{d(P, OB), d(P, AB)\}$ , when  $d$  denotes the distance from the point to the corresponding line. Sketch the region  $R$  and find its area. (IIT 1997C; 5M)
- Let  $f(x) = \text{maximum}\{x^2, (1-x)^2, 2x(1-x)\}$   
where  $0 \leq x \leq 1$ . Determine the area of the region bounded by the curves  $y = f(x)$ ,  $x$ -axis,  $x = 0$  and  $x = 1$ . (IIT 1997; 5M)
- Let  $C_1$  and  $C_2$  be the graphs of functions  $y = x^3$  and  $y = 2x$ ,  $0 \leq x \leq 1$  respectively. Let  $C_3$  be the graph of a function  $y = f(x)$ ,  $0 \leq x \leq 1$ ,  $f(0) = 0$ . For a point  $P$  on  $C_1$ , let the lines through  $P$ , parallel to the axes, meet  $C_2$  and  $C_3$  at  $Q$  and  $R$  respectively (see figure). If for every position of  $P$  (on  $C_1$ ) the areas of the shaded regions  $OPQ$  and  $ORP$  are equal, determine the function  $f(x)$ . (IIT 1998; 8M)



- Let  $f(x)$  be a continuous function given by

$$f(x) = \begin{cases} 2x, & |x| \leq 1 \\ x^2 - ax - b, & |x| > 1 \end{cases}$$

Find the area of the region in the third quadrant bounded by the curves  $x = -2y^2$  and  $y = f(x)$  lying on the left on the line  $8x + 1 = 0$ .

- Let  $b \neq 0$  and for  $j = 0, 1, 2, \dots, n$ , let  $S_j$  be the area of the region bounded by the  $y$ -axis and the curve  $xe^{by} = \sin by$ ,  $\frac{j\pi}{b} \leq y \leq \frac{(j+1)\pi}{b}$ . Show that  $S_0, S_1, S_2, \dots, S_n$  are in geometric progression. Also, find their sum for  $a = -1$  and  $b = \pi$ . (IIT 2001; 5M)

21. Find the area of the region bounded by the curves  $y = x^2$ ,  $y = |2 - x^2|$  and  $y = 2$ , which lies to the right of the line  $x = 1$ ? (IIT 2002; 5M)
22. A curve passes through  $(2, 0)$  and the slope of tangent at point  $P(x, y)$  equals  $\frac{(x+1)^2 + y - 3}{(x+1)}$ . Find the equation of the curve and area enclosed by the curve and the  $x$ -axis in the fourth quadrant. (IIT 2004)
23. The area of the triangle formed by the intersection of a line parallel to  $x$ -axis and passing through  $(h, k)$  with the lines  $y = x$  and  $x + y = 2$  is  $4h^2$ . Find the locus of point  $P$ . (IIT 2005)

24. Find the area bounded by the curves  $x^2 = y$ ,  $x^2 = -y$  and  $y^2 = 4x + 3$ .
25. If  $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$ ,  $f(x)$  is a quadratic function and its maximum value occurs at a point  $V$ .  $A$  is a point of intersection of  $y = f(x)$  with  $x$ -axis and point  $B$  is such that chord  $AB$  subtends a right angle at  $V$ . Find the area enclosed by  $f(x)$  and chord  $AB$ . (IIT 2005)

## ANSWERS

### A Fill In the Blanks

1.  $\sqrt{2}$  square units      2.  $2\sqrt{3}$  square units

### C Objective Questions (Only one option)

1. (c)      2. (c)      3. (b)      4. (d)      5. (a)      6. (a)      7. (a)
8. (a)

### D Objective Questions (More than one option)

1. (b, d)

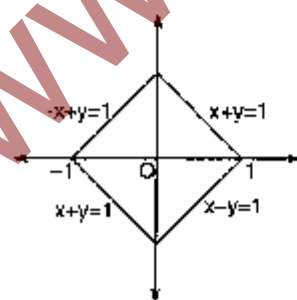
### E Subjective Questions

1.  $\frac{9}{8}$  sq. units      2.  $2\sqrt{2}$       3.  $\frac{1}{2} \log_e 3$       4.  $\frac{5\pi}{4} - \frac{1}{2}$       5.  $\frac{1}{3} - \pi$       6.  $8 + \left(\frac{25\pi}{4}\right)$       7.  $\left(\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \log 2\right)$  sq. units
8.  $y_{\max} = \frac{4}{27}$ ,  $y_{\min} = 0, \frac{2}{3}$  sq. units      9.  $\frac{e^2 - 5}{4e}$       10.  $\frac{4 - \sqrt{2}}{\log_2} - \frac{5}{2} \log 2 + \frac{3}{2}$       11.  $\pi - \frac{2}{3}$       12. 121 : 4
13.  $\frac{1}{3}(16\sqrt{2} - 20)$       15.  $b - 1$       16.  $(2 - \sqrt{3})$  sq. units      17.  $\frac{17}{27}$  sq. units      18.  $f(x) = x^3 - x^2, 0 \leq x \leq 1$
19.  $\frac{761}{192}$       20.  $\frac{\pi(1+e)}{(1+\pi^2)} \cdot \frac{(e^{\pi+1} - 1)}{e - 1}$       21.  $\left(\frac{20 - 12\sqrt{2}}{3}\right)$  sq. units      22.  $\frac{4}{3}$  sq. units,  $y = x^2 - 2x$       23.  $2x = \pm(y - 1)$
24.  $\frac{1}{3}$  sq. units      25.  $\frac{125}{3}$  sq. units

## SOLUTIONS

### A FILL IN THE BLANKS

1. The area formed by  $|x| + |y| = 1$  is square shown as,



whose area =  $(\sqrt{2})^2$   
= 2 square units

2. Equation of tangent to

$$x^2 - y^2 = 4$$

is

$$x + \sqrt{3}y = 4$$

whose  $x$ -axis intercept  $(4, 0)$ .

Thus, area of  $\Delta$  formed by  $(0, 0)$ ,  $(1, \sqrt{3})$  and  $(4, 0)$  is,

$$\frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 1 & \sqrt{3} & 1 \\ 4 & 0 & 1 \end{vmatrix} = \frac{1}{2} |(0 - 4\sqrt{3})| = 2\sqrt{3} \text{ square units}$$

**C OBJECTIVE (ONLY ONE OPTION)**

1. Since,  $\int_1^b f(x) dx = (b-1) \sin(3b+4)$

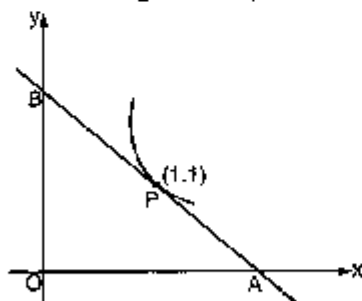
∴ differentiating both sides with respect to  $b$ , we get

$$f(b) = 3(b-1) \cdot \cos(3b+4) + \sin(3b+4)$$

$$\therefore f(x) = \sin(3x+4) + 3(x-1) \cos(3x+4)$$

2.  $f(x) = x^2 + bx - b$

For equation of the tangent at  $P(1, 1)$  is



$$y = x^2 + bx - b$$

$$\Rightarrow 2y = 2x^2 + 2bx - 2b$$

$$\Rightarrow y + 1 = 2x + b(x+1) - 2b \quad (\text{apply } T=0)$$

$$\Rightarrow y = (2+b)x - (1+b)$$

$$\therefore x_A = \frac{1+b}{2+b} \quad \text{and} \quad y_B = -(1+b)$$

$$\text{Again area of } \triangle OAB = \frac{1}{2} OA \times OB = -\frac{1}{2} \times \frac{(1+b)^2}{(2+b)} = 2$$

(given)

$$\Rightarrow -(1+b)^2 = 4(2+b)$$

$$\Rightarrow (1+b)^2 - 4(2+b) = 0$$

$$\Rightarrow 1 + b^2 + 2b + 8 + 4b = 0$$

$$\Rightarrow b^2 + 6b + 9 = 0$$

$$\Rightarrow (b+3)^2 = 0$$

$\Rightarrow b = -3$ , therefore, (c) is the answer.

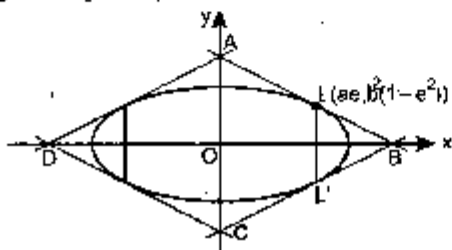
3. The region is clearly square with vertices at the points  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$  and  $(0, -1)$ . So that its area  $= \sqrt{2} \times \sqrt{2} = 2$ .

4. Given:  $\frac{x^2}{9} + \frac{y^2}{5} = 1$

To find tangents at the end points of latus rectum, we find  $ae$ .

$$\text{i.e., } ae = \sqrt{a^2 - b^2} = \sqrt{4} = 2$$

By symmetry the quadrilateral is rhombus.



So area is four times the area of the right angled  $\Delta$  formed by the tangent and axes in the 1st quadrant.

$\Rightarrow$  Equation of tangent at (given)

$$(ae, b^2 (1 - e^2)) = \left(2, \frac{5}{3}\right) \text{ is}$$

$$\frac{2}{9}x + \frac{5}{3} \cdot \frac{y}{5} = 1$$

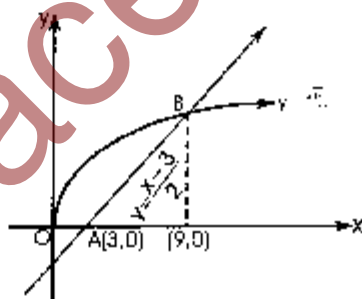
$$\Rightarrow \frac{x}{9/2} + \frac{y}{3} = 1$$

∴ Area of quadrilateral ABCD

$$= 4(\text{area of } \triangle AOB)$$

$$= 4 \cdot \left\{ \frac{1}{2} \cdot \frac{9}{2} \cdot 3 \right\} = 27 \text{ sq. units}$$

5. To find the area between the curves;  $y = \sqrt{x}$  and  $2y + 3 = x$  and  $x$ -axis in the 1st quadrant (we can plot the above condition as);



To find area of shaded portion OABO

$$f(x) = \int_0^9 \sqrt{x} dx - \int_3^9 \left( \frac{x-3}{2} \right) dx$$

$$= \left( \frac{x^{3/2}}{3/2} \right)_0^9 - \frac{1}{2} \left( \frac{x^2}{2} - 3x \right)_3^9$$

$$= \left( \frac{2}{3} \cdot 27 \right) - \frac{1}{2} \left\{ \left( \frac{81}{2} - 27 \right) - \left( \frac{9}{2} - 9 \right) \right\}$$

$$= 18 - \frac{1}{2} \{18\} = 9 \text{ sq. units}$$

6. As from the figure, area enclosed between curves is OABCO.

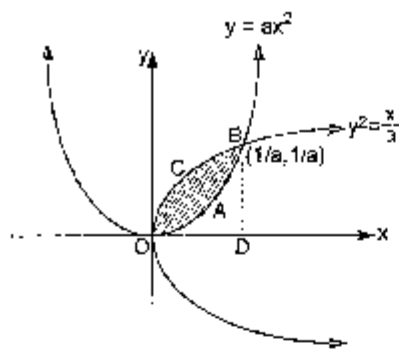
Thus the point of intersection of

$$y = ax^2 \quad \text{and} \quad x = ay^2$$

$$\Rightarrow x = a(ax^2)$$

$$\Rightarrow x = 0, \frac{1}{a}$$

∴ Point of intersection  $(0, 0)$  and  $\left(\frac{1}{a}, \frac{1}{a}\right)$



Thus required area  $OABCO = \text{Area of } OCBD - \text{area of } OABD$

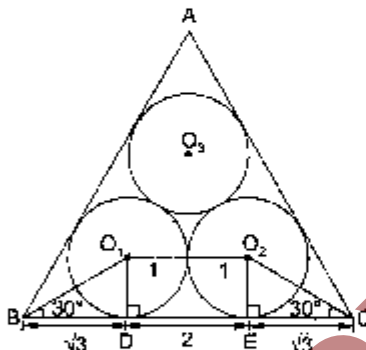
$$\Rightarrow \int_0^{1/a} \left( \sqrt{\frac{x}{3}} - ax^2 \right) dx = 1 \quad (\text{given})$$

$$\therefore \left( \frac{1}{\sqrt{3}} \cdot \frac{x^{3/2}}{3/2} - \frac{ax^3}{3} \right) \Big|_0^{1/a} = 1$$

$$\Rightarrow \frac{2}{3a^2} - \frac{1}{3a^2} = 1$$

$$\Rightarrow a^2 = \frac{1}{3} \text{ or } a = \frac{1}{\sqrt{3}} \quad (\text{as } a > 0)$$

7. As tangents drawn from external point to the circle subtend equal angle at centre.



$$\therefore \angle O_1BD = 30^\circ$$

$$\text{In } \triangle O_1BD, \quad \frac{O_1D}{BD} = \tan 30^\circ$$

$$\Rightarrow BD = 1 \cdot \sqrt{3}$$

#### **D** OBJECTIVE (MORE THAN ONE OPTION)

1. Case 1.  $m = 0$

$$\text{In this case } y = x - x^2 \quad \dots(1)$$

$$y = 0 \quad \dots(2)$$

are two given curves,  $y > 0$  is total region above  $x$ -axis.

Therefore, area between  $y = x - x^2$  and  $y = 0$

is area between  $y = x - x^2$  and above the  $x$ -axis

$$A = \int_0^1 (x - x^2) dx = \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \neq \frac{9}{2}$$

Also,  $DE = O_1O_2 = 2$  and  $EC = \sqrt{3}$

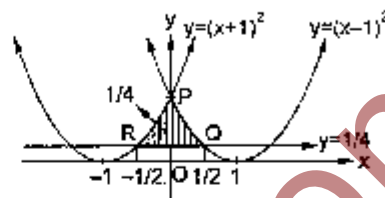
Hence,

$$BC = BD + DE - EC = 2 - 2\sqrt{3}$$

$$\Rightarrow \text{Area} = \frac{\sqrt{3}}{4} (BC)^2 = \frac{\sqrt{3}}{4} \cdot 4(1 + \sqrt{3})^2$$

$$= 6 + 4\sqrt{3} \text{ square cm.}$$

8. The curves  $y = (x-1)^2$ ,  $y = (x+1)^2$  and  $y = 1/4$  are shown as :



where point of intersection are

$$(x-1)^2 = \frac{1}{4}$$

$$\Rightarrow x = \frac{1}{2} \text{ or } \frac{1}{2}$$

$$\therefore Q\left(\frac{1}{2}, \frac{1}{4}\right) \text{ and } R\left(-\frac{1}{2}, \frac{1}{4}\right)$$

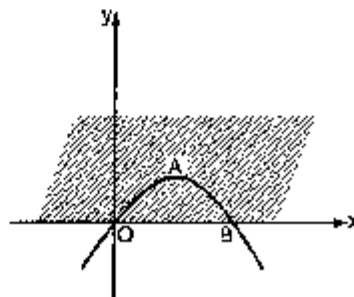
$$\Rightarrow \text{Area} = 2 \int_0^{1/2} \left\{ (x-1)^2 - \frac{1}{4} \right\} dx$$

$$= 2 \left[ \frac{(x-1)^3}{3} - \frac{1}{4}x \right] \Big|_0^{1/2}$$

$$= 2 \left[ \frac{(-1/2)^3}{3} - \frac{1}{8} - \left( -\frac{1}{3} - 0 \right) \right]$$

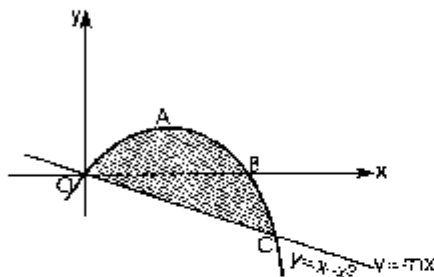
$$= \frac{8}{24} = \frac{1}{3} \text{ square units}$$

Hence no solution.



**Case 2.**  $m < 0$

In this case area between



$$y = x - x^2 \text{ and } y = mx \text{ is}$$

$OABCO$  and points of intersection are  $(0,0)$  and  $\{(1-m), m(1-m)\}$

$$\text{Area } OABCO = \int_0^{1-m} [x - x^2 - mx] dx.$$

**Imp. note :** Area  $OBCO$  considered automatically because  $m$  is a parameter

$$\begin{aligned} &= \left[ (1-m) \frac{x^2}{2} - \frac{x^3}{3} \right]_0^{1-m} \\ &= \frac{1}{2} (1-m)^3 - \frac{1}{3} (1-m)^3 = \frac{1}{6} (1-m)^3 \end{aligned}$$

Put Area  $OABCO = 9/2$  (given)

$$\therefore \frac{1}{6} (1-m)^3 = \frac{9}{2}$$

$$\Rightarrow (1-m)^3 = 27$$

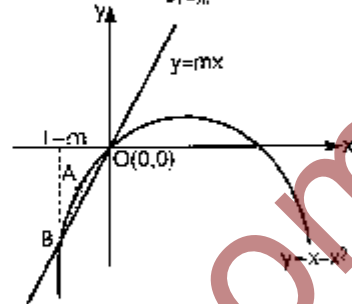
$$\Rightarrow 1-m = 3$$

$$\Rightarrow m = -2$$

**Case 3.**  $m > 0$

In this case  $y = mx$  and  $y = x - x^2$  intersect in  $(0,0)$  and  $\{(1-m), m(1-m)\}$  as shown in Fig.

$$\text{Area of shaded region} = \int_{1-m}^0 (x - x^2 - mx) dx$$



$$\begin{aligned} &= \left[ (1-m) \frac{x^2}{2} - \frac{x^3}{3} \right]_{1-m}^0 \\ &= -\frac{1}{2} (1-m) (1-m)^2 + \frac{1}{3} (1-m)^3 \\ &= -\frac{1}{6} (1-m)^3 \end{aligned}$$

Area of shaded region =  $9/2$  sq. unit (given)

$$\Rightarrow \frac{9}{2} = -\frac{1}{6} (1-m)^3$$

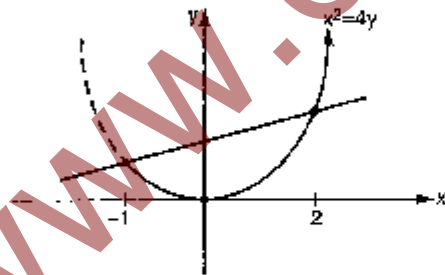
$$\Rightarrow (1-m)^3 = -27$$

$$\Rightarrow (1-m) = -3$$

$\Rightarrow m = 3 + 1 = 4$ . Therefore (b) and (d) are the answers.

### E SUBJECTIVE QUESTIONS

1. The curves  $x^2 = 4y$  and  $x = 4y - 2$  could be sketched as, whose point of intersection are  $x = -1$ , and  $x = 2$



$$\begin{aligned} \text{Thus, area} &= \int_{-1}^2 \left\{ \left( \frac{x+2}{4} \right) - \left( \frac{x^2}{4} \right) \right\} dx \\ &= \frac{1}{4} \left\{ \frac{x^2}{2} + 2x - \frac{x^3}{3} \right\}_{-1}^2 \\ &= \frac{1}{4} \left\{ \left( 2 + 4 - \frac{8}{3} \right) - \left( \frac{1}{2} - 2 + \frac{1}{3} \right) \right\} \end{aligned}$$

$$= \frac{1}{4} \cdot \frac{9}{2} = \frac{9}{8} \text{ sq. units}$$

$$2. \text{ Here, } \int_2^a \left( 1 + \frac{8}{x^2} \right) dx = \int_2^a \left( 1 + \frac{8}{x^2} \right) dx$$

$$\Rightarrow \left( x - \frac{8}{x} \right)_2^a = \left( x - \frac{8}{x} \right)_2^a$$

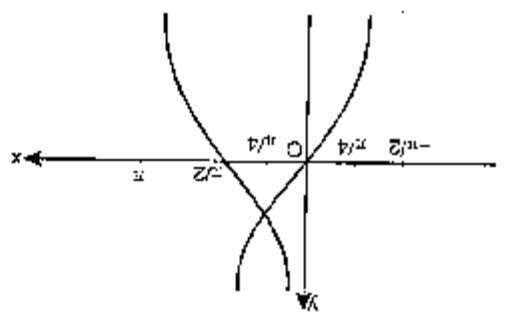
$$\Rightarrow \left( a - \frac{8}{a} \right) - (2 - 4) = (4 - 2) - \left( a - \frac{8}{a} \right)$$

$$\Rightarrow a - \frac{8}{a} + 2 = 2 - a + \frac{8}{a}$$

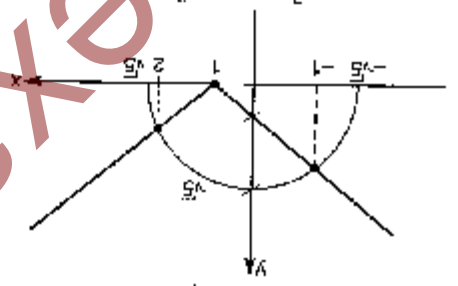
$$\Rightarrow 2a - \frac{16}{a} = 0 \text{ or } 2(a^2 - 8) = 0$$

$$\Rightarrow a = \pm 2\sqrt{2}, \text{ neglecting } (-ve) \\ a = 2\sqrt{2}.$$

3. As  $y = \begin{cases} \tan x, & \frac{3}{\pi} \leq x \leq \frac{3}{2} \\ \cot x, & \frac{1}{\pi} \leq x \leq \frac{6}{\pi} \end{cases}$  which could be plotted as, y-axis.

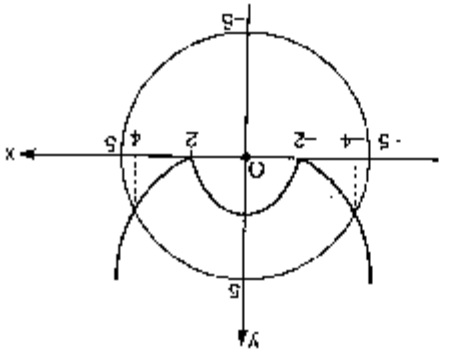


Required area =  $\int_{\pi/4}^{\pi/2} (\tan x) dx + \int_{\pi/2}^{3\pi/4} (\cot x) dx$   
 $= (-\log |\cos x|)_{\pi/4}^{\pi/2} + (\log |\sin x|)_{\pi/2}^{3\pi/4}$   
 $= -\left\{ \log \frac{1}{\sqrt{2}} - 0 \right\} + \left\{ \log \frac{\sqrt{3}}{2} - \log \frac{\sqrt{2}}{1} \right\}$   
 $= \log \frac{\sqrt{3}}{2} - 2 \log \frac{\sqrt{2}}{1}$   
 $= \log \frac{\sqrt{3}}{2} - \log \frac{2}{1} = \log \frac{\sqrt{3}}{4} = \frac{1}{2} \log \frac{3}{16}$   
 4.  $y = \sqrt{5-x^2}$  and  $y = |x-1|$  could be sketched as:

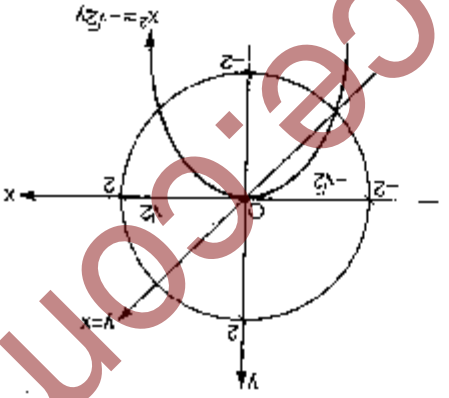


5.  $x^2 = (x-1)^2$   
 $5 - x^2 = x^2 - 2x + 1$   
 $2x^2 - 2x - 4 = 0$   
 $x^2 - x - 2 = 0$   
 $(x-2)(x+1) = 0$   
 $x = 2, -1$   
 $\therefore$  Required area =  $\int_{-1}^2 \sqrt{5-x^2} dx - \int_{-1}^2 (x-1) dx$   
 $= \left[ \frac{x}{2} \sqrt{5-x^2} + \frac{5}{2} \sin^{-1} \left( \frac{x}{\sqrt{5}} \right) \right]_{-1}^2 - \left[ \frac{x^2}{2} - x \right]_{-1}^2$   
 $= \left( 1 + \frac{2}{5} \sin^{-1} \frac{2}{\sqrt{5}} \right) - \left( -1 + \frac{2}{5} \sin^{-1} \frac{-1}{\sqrt{5}} \right) - \left( \frac{4}{2} - 2 - \left( \frac{1}{2} - 1 \right) \right)$

6. Here,  $x^2 + y^2 - 2x - 4y = 4 - x^2$  could be sketched as, whose point of intersection could be obtained by



Thus, the required area =  $\left| \int_{-1}^2 \sqrt{4-x^2} dx \right| - \left| \int_{-1}^2 x dx \right| + \left| \int_{-1}^2 \frac{\sqrt{5-x^2}}{2} dx \right|$   
 $= 2 \int_{-1}^2 \sqrt{4-x^2} dx - \left[ \frac{x^2}{2} \right]_{-1}^2 + \left[ \frac{x}{2} \sqrt{4-x^2} - \frac{2}{4} \sin^{-1} \frac{x}{2} \right]_{-1}^2$   
 $= (2-\pi) - \frac{3}{2} = \frac{1}{2} - \pi$



5. Curves  $x^2 + y^2 = 4$ ,  $x^2 = -\sqrt{2}y$  and  $x = y$  could be shown as.

$\left( \frac{1}{2} + 1 + \frac{1}{2} + 1 \right) - \left( 2 - 2 \frac{1}{2} - 1 \right)$   
 $= 5 \left( \sin^{-1} \frac{1}{2} + \sin^{-1} \frac{1}{1} \right) - \frac{2}{1}$   
 $= \frac{2}{5} \sin^{-1} \left( \frac{\sqrt{5}}{2} \sqrt{\frac{1}{4} - \frac{1}{5}} + \frac{1}{\sqrt{5}} \sqrt{\frac{1}{4} - \frac{1}{5}} \right) - \frac{2}{1}$   
 $= \frac{2}{5} \sin^{-1} (1) - \frac{2}{1} = \frac{4}{2} - \frac{2}{1}$

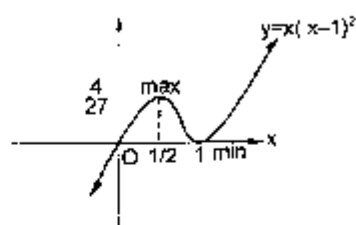
$$\Rightarrow x = \pm 4$$

∴ Required area

$$= 2 \left\{ \int_0^4 \sqrt{25-x^2} dx - \int_0^2 \left( \frac{4-x^3}{4} \right) dx - \int_2^4 \left( \frac{x^3-4}{4} \right) dx \right\}$$

$$= 2 \left\{ \left[ \frac{x}{2} \sqrt{25-x^2} + \frac{25}{2} \sin^{-1} \left( \frac{x}{5} \right) \right]_0^4 - \frac{1}{4} \left( 4x - \frac{x^3}{3} \right) \Big|_0^2 - \frac{1}{4} \left( \frac{x^3}{3} - 4x \right) \Big|_2^4 \right\}$$

$$= 8 + \left( \frac{25\pi}{4} \right)$$



$$= (x-1)(3x-1)$$

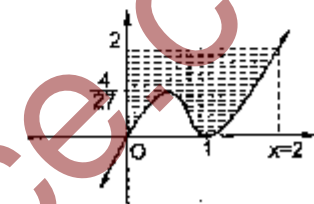
$$\frac{1}{1/3} = 1$$

$$\therefore \text{maximum at } x = 1/3 \Rightarrow y_{\max} = \frac{1}{3} \left( \frac{2}{3} \right)^2 = \frac{4}{27}$$

$$\text{minimum at } x = 1 \Rightarrow y_{\min} = 0$$

Now, to find the area bounded by the curve  $y = x(x-1)^2$ ,

The  $y$ -axis and line  $x = 2$



$$\Rightarrow \text{Area} = \int_0^2 x(x-1)^2 \cdot dx$$

$$= \int_0^2 (x^3 - 2x^2 + x) dx$$

$$= \left[ \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^2$$

$$= \left( 4 - \frac{16}{3} + 2 \right) = 6 - \frac{16}{3} = \frac{2}{3} \text{ sq. units}$$

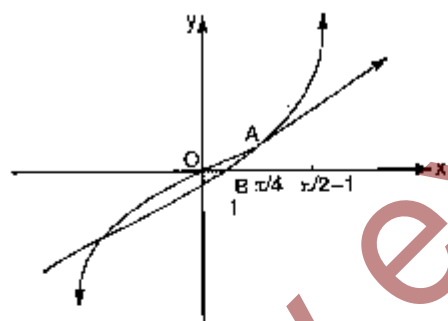
7.  $y = \tan x$

$$\Rightarrow \frac{dy}{dx} = \sec^2 x$$

$$\therefore \left( \frac{dy}{dx} \right)_{x = \pi/4} = 2$$

Hence equation of tangent at  $\left( \frac{\pi}{4}, 1 \right)$  is

$$\frac{y-1}{x-\pi/4} = 2 \quad \text{or} \quad y-1 = 2x - \frac{\pi}{2}$$



or  $(2x - y) = \left( \frac{\pi}{2} - 1 \right)$

∴ Required area is  $OABO$

$$= \int_0^{\pi/4} (\tan x) dx - \int_0^{\pi/4} \left( 2x + 1 - \frac{\pi}{2} \right) dx$$

$$= (\log |\sec x|) \Big|_0^{\pi/4} - \left( x^2 + x - \frac{\pi}{2} x \right) \Big|_0^{\pi/4}$$

$$= \log \sqrt{2} - \left( \frac{\pi}{4} - \frac{\pi^2}{16} \right) = \left( \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \log 2 \right) \text{ sq. units}$$

8.  $y = x(x-1)^2$

$$\Rightarrow \frac{dy}{dx} = x \cdot 2(x-1) + (x-1)^2$$

$$= (x-1) \cdot (2x-x-1)$$

9. Both the curves are defined for  $x > 0$ .

Both are positive when  $x > 1$  and negative when  $0 < x < 1$ .

We know,  $\lim_{x \rightarrow 0^+} (\log x) \rightarrow -\infty$

Hence,  $\lim_{x \rightarrow 0^+} \frac{\log x}{ex} \rightarrow -\infty$ . Thus,  $y$ -axis is asymptote of second curve.

and  $\lim_{x \rightarrow 0^+} ex \log x$  {(0)  $\times$   $\infty$  form}

$$= \lim_{x \rightarrow 0^+} \frac{e \log x}{1/x} \left( \frac{-\infty}{\infty} \text{ form} \right)$$

$$= \lim_{x \rightarrow 0^+} \frac{e \left( \frac{1}{x} \right)}{\left( -\frac{1}{x^2} \right)} = 0 \quad \text{(using L-Hospital's rule)}$$

Thus, the first curve starts from  $(0, 0)$  but does not include  $(0, 0)$

Now, the given curves intersect therefore

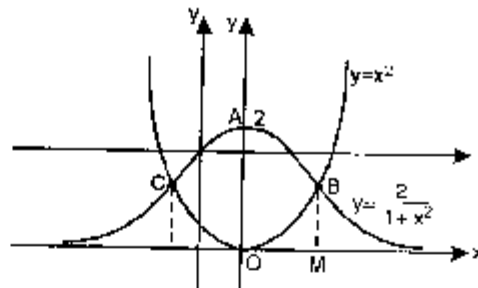
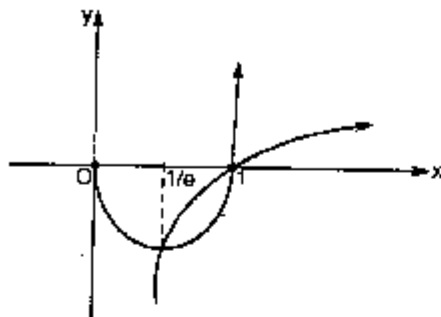
$$ex \log x = \frac{\log x}{ex}$$



i.e.,  $(e^2 x^2 - 1) \log x = 0$

$\Rightarrow x = 1, \frac{1}{e}$  (since  $x > 0$ )

Hence, using above results figure could be drawn as,



To obtain the area, we need point of intersection of

$$y = x^2 \quad \dots(1)$$

and

$$y = \frac{2}{1+x^2} \quad \dots(2)$$

$\Rightarrow$

$$y = \frac{2}{1+y}$$

$\Rightarrow$

$$y + y^2 = 2$$

$\Rightarrow$

$$y^2 + y - 2 = 0$$

$\Rightarrow$

$$y^2 + 2y - y - 2 = 0$$

$\Rightarrow$

$$y(y+2) - 1(y+2) = 0$$

$\Rightarrow$

$$(y-1)(y+2) = 0$$

$y = -2, 1$  but  $y \geq 0$  so  $y = 1 \Rightarrow x = \pm 1$

Therefore coordinates of C are  $(-1, 1)$  and coordinates of B are  $(1, 1)$ .

The required area is  $OBACO = 2$  area of curve  $OBACO$

$$= 2 \left[ \int_0^1 \frac{2}{1+x^2} dx - \int_0^1 x^2 dx \right]$$

$$= 2 \left[ 2 \tan^{-1} x \Big|_0^1 - \left[ \frac{x^3}{3} \right]_0^1 \right]$$

$$= 2 \left[ \frac{2\pi}{4} - \frac{1}{3} \right] = \pi - \frac{2}{3}$$

12.

$$y = 4x - x^2$$

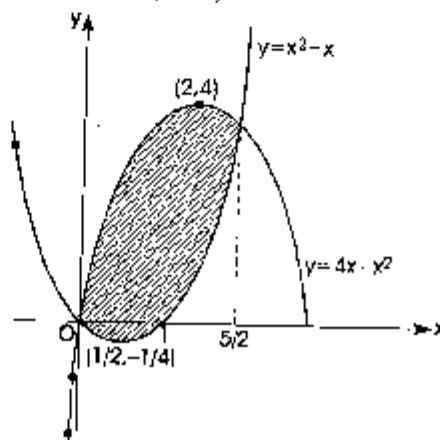
(given)

$$= -(x^2 - 4x + 4 - 4)$$

$$= -(x^2 - 4x + 4) + 4$$

$$y = -(x-2)^2 + 4$$

$$y - 4 = -(x-2)^2$$



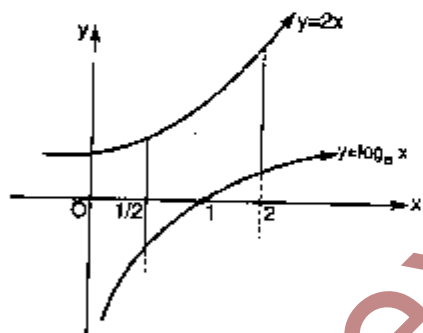
$\therefore$  The required area

$$= \int_{1/e}^1 \left( \frac{(\log x)}{ex} - ex \log x \right) dx$$

$$= \frac{1}{e} \left\{ \frac{(\log x)^2}{2} \right\}_{1/e}^1 - e \left\{ \frac{x^2}{4} (2 \log x - 1) \right\}_{1/e}^1$$

$$= \frac{e^2 - 5}{4e}$$

10. The required area is the shaded portion in following figure.



In the region  $\frac{1}{2} \leq x \leq 2$  the curve  $y = 2x$  lies above as

compared to  $y = \log_e x$

Hence, the required area

$$= \int_{1/2}^2 (2x - \log x) dx$$

$$= \left( \frac{2x^2}{2} - (x \log x - x) \right)_{1/2}^2$$

$$= \frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2} \log 2 - \frac{3}{2}$$

11. The curve  $y = x^2$  is a parabola. It is symmetric about

$y$ -axis and has its vertex at  $(0, 0)$  and the curve  $y = \frac{2}{1+x^2}$

is a bell shaped curve.  $x$ -axis is its asymptote and it is symmetric about  $y$ -axis and its vertex is  $(0, 2)$ .

Therefore it is a vertically downward parabola with vertex at (2,4) and its axis is  $x = 2$

and  $y = x^2 - x$  (given)

$$\Rightarrow y = x^2 - x + \frac{1}{4} - \frac{1}{4}$$

$$\Rightarrow y = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$$

$$\Rightarrow y + \frac{1}{4} = (x - 1/2)^2$$

This is a parabola having its vertex at  $\left(\frac{1}{2}, -\frac{1}{4}\right)$

its axis at  $x = \frac{1}{2}$  and opening upwards.

To obtain the  $x$ -coordinate of the points of intersection we solve  $y = 4x - x^2$  and  $y = x^2 - x$

$$\Rightarrow 4x - x^2 = x^2 - x$$

$$\Rightarrow 2x^2 = 5x \Rightarrow 2x^2 - 5x = 0 \Rightarrow x(2 - 5x) = 0$$

$$\Rightarrow x = 0, \frac{5}{2}$$

Also  $y = x^2 - x$ , meets  $x$ -axis at (0,0) and (1, 0)

Now area,  $A_1 = \int_0^{5/2} [(4x - x^2)] \cdot [(x^2 - x)] dx$

$$= \int_0^{5/2} (5x - 2x^2) dx$$

$$= \left[ \frac{5}{2}x^2 - \frac{2}{3}x^3 \right]_0^{5/2}$$

$$= \frac{5}{2} \left(\frac{5}{2}\right)^2 - \frac{2}{3} \left(\frac{5}{2}\right)^3$$

$$= \frac{5}{2} \cdot \frac{25}{4} - \frac{2}{3} \cdot \frac{125}{8}$$

$$= \frac{125}{8} \left(1 - \frac{2}{3}\right) = \frac{125}{24}$$

This area is considering above and below  $x$ -axis both. Now for area below  $x$ -axis separately. We consider

$$A_2 = - \int_0^1 (x^2 - x) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Therefore net area above the  $x$ -axis

$$= A_1 - A_2 = \frac{125}{24} - \frac{1}{6} = \frac{121}{24}$$

Hence, ratio of area above the  $x$ -axis and area below  $x$ -axis

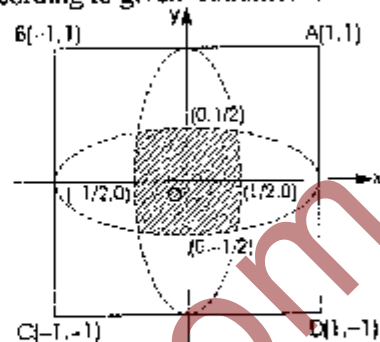
$$= \frac{121}{24} : \frac{1}{6}$$

$$= 121 : 4$$

13. The equations of the sides of the square are as follows :

$$AB: y = 1, BC: x = -1, CD: y = -1, DA: x = 1$$

Let the region be  $S$  and  $(x, y)$  is any point inside it. Then according to given conditions,



$$\sqrt{x^2 + y^2} < |1 - x|, |1 + x|, |1 - y|, |1 + y|$$

$$\Rightarrow x^2 + y^2 < (1 - x)^2, (1 + x)^2, (1 - y)^2, (1 + y)^2$$

$$\Rightarrow x^2 + y^2 < x^2 - 2x + 1, x^2 + 2x + 1,$$

$$y^2 - 2y + 1, y^2 + 2y + 1$$

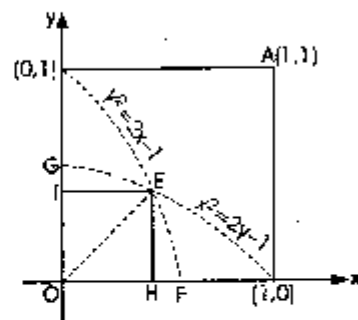
$$\Rightarrow y^2 < 1 - 2x, y^2 < 1 + 2x, x^2 < 1 - 2y \text{ and } x^2 < 1 + 2y$$

Now in  $y^2 = 1 - 2x$  and  $y^2 = 1 + 2x$ , the first equation represents a parabola with vertex at  $(1/2, 0)$  and second equation represents a parabola with vertex  $(-1/2, 0)$ .

And in  $x^2 = 1 - 2y$  and  $x^2 = 1 + 2y$ , the first equation represents a parabola with vertex at  $(0, 1/2)$  and second equation represents a parabola with vertex at  $(0, -1/2)$ .

Therefore, the region  $S$  is the region lying inside the four parabolas

$$y^2 = 1 - 2x, y^2 = 1 + 2x, x^2 = 1 + 2y, x^2 = 1 - 2y$$



where  $S$  is the shaded region.

Now,  $S$  is symmetrical in all four quadrants, therefore,  $S = 4 \times$  area lying in the first quadrant.

Now,  $y^2 = 1 - 2x$  and  $x^2 = 1 - 2y$  intersect on the line  $y = x$ . The point of intersection is  $E(\sqrt{2} - 1, \sqrt{2} - 1)$

Area of the region  $OEFO$

$$= \text{area of } \triangle OEH + \text{area of } HEFH$$

$$= \frac{1}{2}(\sqrt{2} - 1)^2 + \int_{\sqrt{2}-1}^{1/2} \sqrt{1 - 2x} dx$$

$$\begin{aligned}
 &= \frac{1}{2}(\sqrt{2}-1)^2 + \left[ (1-2x)^{3/2} \cdot \frac{2}{3} \cdot \frac{1}{2}(-1) \right]_{-\sqrt{2}-1}^{1/2} \\
 &= \frac{1}{2}(2+1-2\sqrt{2}) + \frac{1}{3}(1+2-2\sqrt{2})^{3/2} \\
 &= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}(3-2\sqrt{2})^{3/2} \\
 &= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}[(\sqrt{2}-1)^2]^{3/2} \\
 &= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}(\sqrt{2}-1)^3 \\
 &= \frac{1}{2}(3-\sqrt{2}) + \frac{1}{3}[2\sqrt{2}-1-3\sqrt{2}(\sqrt{2}-1)] \\
 &= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}[5\sqrt{2}-7] \\
 &= \frac{1}{6}[9-6\sqrt{2}+10\sqrt{2}-14] = \frac{1}{6}[4\sqrt{2}-5]
 \end{aligned}$$

Similarly, area  $OEGO = \frac{1}{6}(4\sqrt{2}-5)$

Therefore, area of  $S$  lying in first quadrant

$$\begin{aligned}
 &= \frac{2}{6}(4\sqrt{2}-5) \\
 &= \frac{1}{3}(4\sqrt{2}-5)
 \end{aligned}$$

Hence,  $S = \frac{4}{3}(4\sqrt{2}-5) = \frac{1}{3}(16\sqrt{2}-20)$

14. We have  $A_n = \int_0^{\pi/4} (\tan x)^n dx$

Since  $0 < \tan x < 1$ , when  $0 < x < \pi/4$  we have

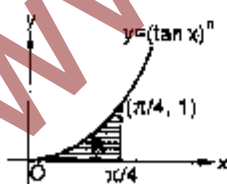
$$0 < (\tan x)^{n+1} < (\tan x)^n \text{ for each } n \in \mathbb{N}$$

$$\Rightarrow \int_0^{\pi/4} (\tan x)^{n+1} dx < \int_0^{\pi/4} (\tan x)^n dx$$

$$\Rightarrow A_{n+1} < A_n$$

Now, for  $n > 2$

$$\begin{aligned}
 A_n + A_{n+2} &= \int_0^{\pi/4} [(\tan x)^n + (\tan x)^{n+2}] dx \\
 &= \int_0^{\pi/4} (\tan x)^n (1 + \tan^2 x) dx
 \end{aligned}$$



$$= \int_0^{\pi/4} (\tan x)^n \sec^2 x dx$$

$$= \left[ \frac{1}{(n+1)} (\tan x)^{n+1} \right]_0^{\pi/4}$$

$$\left[ \int f(x)^n f'(x) dx = \frac{f(x)^{n+1}}{n+1} \right]$$

$$= \frac{1}{(n+1)}(1-0) = \frac{1}{n+1}$$

Since  $A_{n+2} < A_{n+1} < A_n$ , we get

$$A_n + A_{n+2} < 2A_{n+1}$$

$$\Rightarrow \frac{1}{n+1} < 2A_n$$

$$\Rightarrow \frac{1}{2n+2} < A_n \quad \dots(1)$$

Also for  $n > 2$

$$A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$$

$$\Rightarrow 2A_n < \frac{1}{n-1}$$

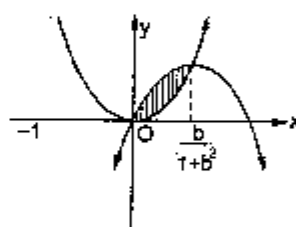
$$\Rightarrow A_n < \frac{1}{2n-2} \quad \dots(2)$$

from (1) and (2)  $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$

15. Eliminating  $y$  from  $y = \frac{x^2}{b}$  and  $y = x - bx^2$ , we get

$$x^2 = bx - b^2x^2$$

$$\Rightarrow x = 0, \frac{b}{1+b^2}$$



Thus, the area enclosed between the parabolas.

$$A = \int_0^{b/(1+b^2)} \left( x - bx^2 - \frac{x^2}{b} \right) dx$$

$$= \left[ \frac{x^2}{2} - \frac{x^3}{3} - \frac{1+b^2}{b} \cdot \frac{x^3}{3} \right]_0^{b/(1+b^2)}$$

$$= \frac{1}{6} \cdot \frac{b^2}{(1+b^2)^2}$$

for maximum, value of  $A$ ,  $\frac{dA}{db} = 0$

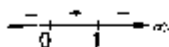
$$\text{But } \frac{dA}{db} = \frac{1}{6} \cdot \frac{(1+b^2)^2 \cdot 2b - 2b^2 \cdot (1+b^2) \cdot 2b}{(1+b^2)^4}$$

$$= \frac{1}{3} \cdot \frac{b(1-b^2)}{(1+b^2)^3}$$

Hence,  $\frac{dA}{db} = 0$  gives  $b = -1, 0, 1$  since  $b > 0$

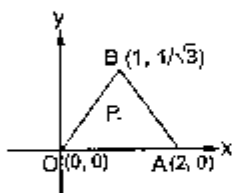
$\therefore$  we consider only  $b = 1$ .

Sign scheme for  $\frac{dA}{db}$  around  $b=1$  is as below



from sign scheme it is clear  $A$  is maximum at  $b=1$ .

16. Let the coordinate of  $P$  be  $(x, y)$



Equation of line  $OA$  be  $y=0$

Equation of line  $OB$  be  $\sqrt{3}y=x$

Equation of line  $AB$  be  $\sqrt{3}y=2-x$

$d(P, OA)$  = distance of  $P$  from line  $OA = y$

$d(P, OB)$  = distance of  $P$  from line  $OB = \frac{|\sqrt{3}y-x|}{2}$

$d(P, AB)$  = distance of  $P$  from line  $AB = \frac{|\sqrt{3}y+x-2|}{2}$

Given :  $d(P, OA) \leq \min. \{d(P, OB), d(P, AB)\}$

$$y \leq \min. \left\{ \frac{|\sqrt{3}y-x|}{2}, \frac{|\sqrt{3}y+x-2|}{2} \right\}$$

$$\Rightarrow y \leq \frac{|\sqrt{3}y-x|}{2} \quad \text{and} \quad y \leq \frac{|\sqrt{3}y+x-2|}{2}$$

**Case I :**  $y \leq \frac{|\sqrt{3}y-x|}{2}$ , (since  $\sqrt{3}y-x < 0$ )

$$\Rightarrow y \leq \frac{x-\sqrt{3}y}{2}$$

$$\Rightarrow (2+\sqrt{3})y \leq x$$

$$\Rightarrow y \leq x \tan 15^\circ$$

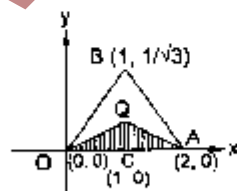
**Case II :** If  $y \leq \frac{|\sqrt{3}y+x-2|}{2}$

$$\Rightarrow 2y \leq 2-x-\sqrt{3}y \quad [\text{i.e., } \sqrt{3}y+x-2 < 0]$$

$$\Rightarrow (2+\sqrt{3})y \leq 2-x$$

$$\Rightarrow y \leq \tan 15^\circ \cdot (2-x)$$

from above discussion  $P$  moves in side the  $\Delta$  as shown.



$\Rightarrow$  Area of shaded region

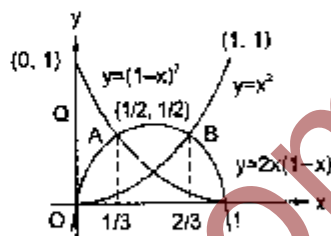
= area of  $\Delta OQA$

$$= \frac{1}{2} (\text{base}) \times \text{height}$$

$$= \frac{1}{2} (2) (\tan 15^\circ) = \tan 15^\circ$$

$$= (2-\sqrt{3}) \text{ sq. units}$$

17. We can draw the graph of  $y=x^2$ ,  $y=(1-x)^2$  and  $y=2x(1-x)$  in following fig. Now, to get the point of intersection of  $y=x^2$  and  $y=2x(1-x)$ . We solve both the equations.



we get

$$x^2 = 2x(1-x)$$

$$\Rightarrow x^2 = 2x - 2x^2$$

$$\Rightarrow 3x^2 = 2x$$

$$\Rightarrow 3x^2 - 2x = 0$$

$$\Rightarrow x(3x-2) = 0$$

$$\Rightarrow x = 0, 2/3$$

Similarly, we can find the coordinate of the points of intersection of

$$y=(1-x)^2 \text{ and } y=2x(1-x) \text{ are } x=1/3 \text{ and } x=1$$

From the figure it is clear that

$$f(x) = \begin{cases} (1-x)^2, & 0 \leq x \leq 1/3 \\ 2x(1-x), & 1/3 \leq x \leq 2/3 \\ x^2, & 2/3 \leq x \leq 1 \end{cases}$$

The required area  $A$  is given by

$$A = \int_0^1 f(x) dx$$

$$= \int_0^{1/3} (1-x)^2 dx + \int_{1/3}^{2/3} 2x(1-x) dx + \int_{2/3}^1 x^2 dx$$

$$= \left[ -\frac{1}{3}(1-x)^3 \right]_0^{1/3} + \left[ x^2 - \frac{2x^3}{3} \right]_{1/3}^{2/3} + \left[ \frac{1}{3}x^3 \right]_{2/3}^1$$

$$= -\frac{1}{3} \left( \frac{2}{3} \right)^3 + \frac{1}{3} + \left( \frac{2}{3} \right)^2 - \frac{2}{3} \left( \frac{2}{3} \right)^3$$

$$- \left( \frac{1}{3} \right)^2 + \frac{2}{3} \left( \frac{1}{3} \right)^2 + \frac{1}{3} (1) - \frac{1}{3} \left( \frac{2}{3} \right)^3$$

$$= \frac{17}{27} \text{ sq. units}$$

18. Refer to the Fig. in the question. Let the coordinates of  $P$  be  $(x, x^2)$ , where  $0 \leq x \leq 1$ .

For the area  $(OPRO)$ , upper boundary  $y=x^2$

lower boundary :  $y = f(x)$

lower limit of  $x: 0$

upper limit of  $x: x$

$$\therefore \text{area (OPRO)} = \int_0^x t^2 dt - \int_0^x f(t) dt$$

$$= \left[ \frac{t^3}{3} \right]_0^x - \int_0^x f(t) dt$$

$$= \frac{x^3}{3} - \int_0^x f(t) dt - \int_0^{x^2} \frac{t}{2} dt$$

For the area (OPQO) the upper curve :  $x = \sqrt{y}$

the lower curve :  $x = y/2$

lower limit of  $y: 0$  and upper limit of  $y: x^2$

$$\therefore \text{area (OPQO)} = \int_0^{x^2} \sqrt{t} dt - \int_0^{x^2} \frac{t}{2} dt$$

$$= \frac{2}{3} [t^{3/2}]_0^{x^2} - \frac{1}{4} [t^2]_0^{x^2}$$

$$= \frac{2}{3} x^3 - \frac{1}{4} x^4$$

according to the given condition,

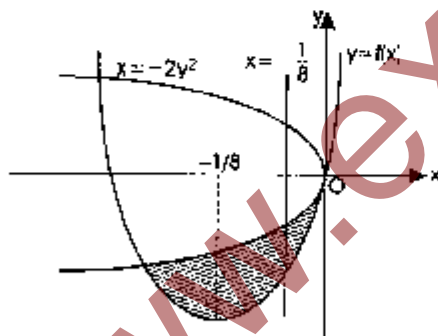
$$\frac{1}{3} x^3 - \int_0^x f(t) dt = \frac{2}{3} x^3 - \frac{x^4}{4}$$

Differentiating both sides w.r.t.  $x$ , we get

$$x^2 - f(x) \cdot 1 = 2x^2 - x^3$$

$$\Rightarrow f(x) = x^3 - x^2, 0 \leq x \leq 1$$

$$19. f(x) = \begin{cases} 2x, & |x| \leq 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$



$$\Rightarrow f(x) = \begin{cases} x^2 + ax + b, & \text{if } x < -1 \\ 2x, & \text{if } -1 \leq x < 1 \\ x^2 + ax + b, & \text{if } 1 \leq x \end{cases}$$

$f$  is continuous on  $R$  so  $f$  is continuous at  $-1$  and  $1$ .

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x) = f(-1)$$

$$\text{and } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

$$\Rightarrow 1 - a + b = -2 \text{ and } 2 = 1 + a + b$$

Thus,  $a - b = 3$  and  $a + b = 1$

$$\therefore a = 2, b = -1$$

$$\text{Hence, } f(x) = \begin{cases} x^2 + 2x - 1 & \text{if } x < -1 \\ 2x & \text{if } -1 \leq x < 1 \\ x^2 + 2x - 1 & \text{if } x \geq 1 \end{cases}$$

Next, we have to find the point  $x = -2y^2$  and  $y = f(x)$

The point of intersection is  $(-2, -1)$

$\therefore$  Required Area

$$= \int_{-2}^{-1/8} \left[ \sqrt{\frac{-x}{2}} - f(x) \right] dx$$

$$= \int_{-2}^{-1/8} \sqrt{\frac{-x}{2}} dx - \int_{-2}^{-1} (x^2 + 2x - 1) dx - \int_{-1}^{-1/8} 2x dx$$

$$= \frac{-2}{3\sqrt{2}} [(-x)^{-1/8}]_{-2}^{-1/8} - \left[ \frac{x^3}{3} + x^2 - x \right]_{-2}^{-1} - [x^2]_{-1}^{-1/8}$$

$$= \frac{-2}{3\sqrt{2}} \left[ \left( \frac{1}{8} \right)^{3/2} - 2^{3/2} \right] - \left( -\frac{1}{3} + 1 + 1 \right) + \left( -\frac{8}{3} + 4 + 2 \right) - \left[ \frac{1}{64} - 1 \right]$$

$$= \frac{\sqrt{2}}{3} [2\sqrt{2} \cdot 2^{-9/2}] + \frac{5}{3} + \frac{63}{64}$$

$$= \frac{761}{192}$$

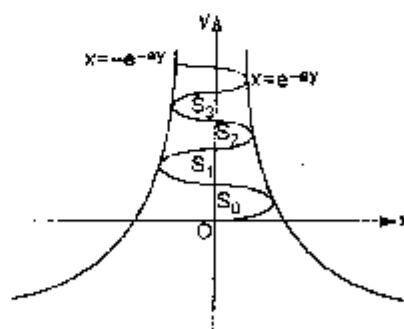
$$20. \text{ Given } x = (\sin by) e^{-ay}$$

Now,

$$-1 \leq \sin by \leq 1$$

$$\Rightarrow -e^{-ay} \leq -e^{-ay} \sin by \leq e^{-ay}$$

$$\Rightarrow -e^{-ay} \leq x \leq e^{-ay}$$



In this case if we take  $a$  and  $b$  positive, the values  $-e^{-ay}$  and  $e^{-ay}$  become left bond and right bond of the curve and due to oscillating nature of  $\sin by$ , it will oscillate between  $x = e^{-ay}$  and  $x = -e^{-ay}$

$$\text{Now, } S_j = \int_{(j\pi/b)}^{(j+1)\pi/b} \sin by \cdot e^{-ay} dy$$

$$\Rightarrow I = \int \sin by \cdot e^{-ay} dy$$

Integrating by parts, we get

$$I = \frac{-e^{-ay}}{a^2 + b^2} (a \sin by + b \cos by)$$

[ by Euler's Integral ]

$$\text{So, } S_j = \left| \frac{-1}{a^2 + b^2} \left[ e^{-\frac{a(j+1)\pi}{b}} \left( a \sin(j+1)\pi + b \cos(j+1)\pi \right) - e^{-\frac{aj\pi}{b}} \left( a \sin j\pi + b \cos j\pi \right) \right] \right|$$

$$S_j = \left| \frac{1}{a^2 + b^2} \left[ e^{-\frac{a(j+1)\pi}{b}} (0 + b(-1)^{j+1}) - e^{-\frac{aj\pi}{b}} (0 - b(-1)^j) \right] \right|$$

$$= \left| \frac{b \cdot (-1)^j e^{-\frac{aj\pi}{b}} \left( e^{-\frac{a\pi}{b}} + 1 \right) \right|$$

$$= \frac{b \cdot e^{-\frac{aj\pi}{b}} \left( e^{-\frac{a\pi}{b}} + 1 \right)}{a^2 + b^2}$$

Now,  $\frac{S_j}{S_{j-1}} = \frac{a^2 + b^2}{b e^{-\frac{a(j-1)\pi}{b}} \left( e^{-\frac{a\pi}{b}} + 1 \right)} \cdot \frac{e^{-\frac{aj\pi}{b}} \left( e^{-\frac{a\pi}{b}} + 1 \right)}{a^2 + b^2}$

$$\Rightarrow \frac{e^{-\frac{aj\pi}{b}}}{e^{-\frac{a(j-1)\pi}{b}}} = e^{-\frac{a\pi}{b}} = \text{constant}$$

$\Rightarrow \frac{S_j}{S_{j-1}} = \text{constant} \Rightarrow S_0, S_1, S_2, \dots, S_j$  form a G.P.

For  $a = -1$  and  $b = \pi$

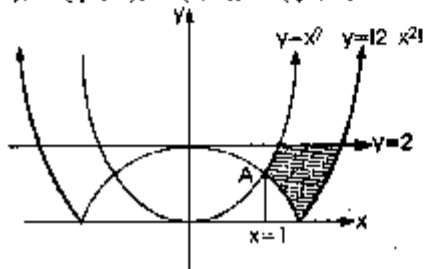
$$S_j = \frac{\pi \cdot e^{\frac{1}{\pi} j \pi} \left( e^{\frac{1}{\pi} \pi} + 1 \right)}{(1 + \pi^2)} = \frac{\pi \cdot j}{(1 + \pi^2)} (1 + e)$$

$$\Rightarrow \sum_{j=0}^n S_j = \frac{\pi(1+e)}{(1+\pi^2)} \sum_{j=0}^n e^j$$

$$= \frac{\pi(1+e)}{(1+\pi^2)} (e^0 + e^1 + \dots + e^n) = \frac{\pi(1+e)}{(1+\pi^2)} \cdot \frac{(e^{n+1} - 1)}{e - 1}$$

21. The points in the graph are :

$A(1, 1), B(\sqrt{2}, 0), C(2, 2), D(\sqrt{2}, 2)$



Required area

$$= \int_1^{\sqrt{2}} (x^2 - (2 - x^2)) dx + \int_{\sqrt{2}}^2 (2 - (x^2 - 2)) dx$$

$$= \int_1^{\sqrt{2}} (2x^2 - 2) dx + \int_{\sqrt{2}}^2 (4 - x^2) dx$$

$$= \left( \frac{2x^3}{3} - 2x \right) \Big|_1^{\sqrt{2}} + \left( 4x - \frac{x^3}{3} \right) \Big|_{\sqrt{2}}^2$$

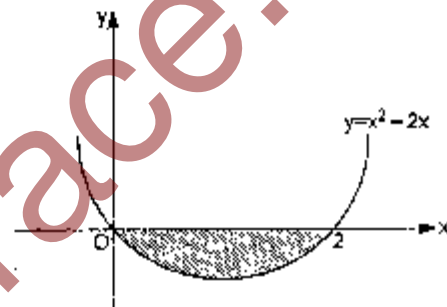
$$= \left( \frac{20 - 12\sqrt{2}}{3} \right) \text{ sq. units.}$$

22. Here, slope of tangent

$$\Rightarrow \frac{dy}{dx} = \frac{(x+1)^2 + y - 3}{(x+1)}$$

$$\rightarrow \frac{dy}{dx} = (x+1) + \frac{(y-3)}{(x+1)}$$

put  $x+1 = X$  and  $y-3 = Y$



$$\Rightarrow \frac{dy}{dx} = \frac{dY}{dX}, \therefore \frac{dY}{dX} = X + \frac{Y}{X} \text{ or } \frac{dY}{dX} - \frac{1}{X} \cdot Y = X$$

where integrating factor

$$= e^{\int -\frac{1}{X} dX} = e^{-\log X} = \frac{1}{X}$$

$\therefore$  Solution is,

$$Y \cdot \frac{1}{X} = \int X \cdot \frac{1}{X} dX + c \text{ or } \frac{Y}{X} = X + c$$

$y-3 = (x+1)^2 + c(x+1)$ , which passes through  $(2, 0)$

$$-3 = 1 + c$$

$$\Rightarrow c = -4$$

$\therefore$  Required curve

$$y = (x+1)^2 - 4(x+1) + 3 \text{ or } y = x^2 - 2x$$

Shown as,

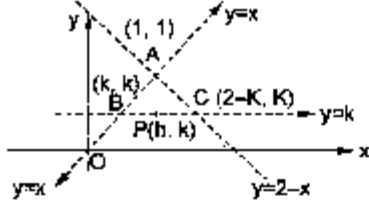
Thus, required area

$$= \left| \int_0^2 (x^2 - 2x) dx \right| = \left| \left( \frac{x^3}{3} - x^2 \right) \Big|_0^2 \right| = \frac{4}{3} \text{ sq. units}$$

23. Here, the triangle formed by a line parallel to  $x$ -axis passing through  $P(h, k)$  and the straight line  $y = x$  and

$y = 2 - x$  could be shown as,

Since, area of  $\Delta ABC = 4h^2$



$$\therefore \frac{1}{2} \cdot AB \cdot AC = 4h^2,$$

where  $AB = \sqrt{2}|k-1|$  and  $AC = \sqrt{2}(k-1)$

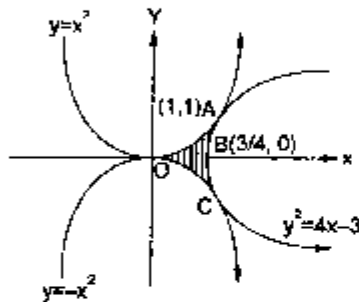
$$\Rightarrow \frac{1}{2} \cdot 2(k-1) = 4h^2$$

$$\therefore 4h^2 = (k-1)^2$$

or  $2h = \pm(k-1)$  is required locus

$$\text{i.e., } 2x = \pm(y-1)$$

24. The region bounded by the curves  $y = x^2$ ,  $y = -x^2$  and  $y^2 = 4x - 3$  is symmetrical about x-axis. where  $y = 4x - 3$  meets at (1, 1)



Hence, area (OABCO)

$$= 2 \left\{ \int_0^1 x^2 dx - \int_{3/4}^1 (\sqrt{4x-3}) dx \right\}$$

$$= 2 \left\{ \left[ \frac{x^3}{3} \right]_0^1 - \left[ \frac{(4x-3)^{3/2}}{32/4} \right]_{3/4}^1 \right\}$$

$$= 2 \left\{ \frac{1}{3} - \frac{1}{6} \right\} = 1 \cdot \frac{1}{6} = \frac{1}{3} \text{ square units}$$

25. Given, 
$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + a \\ 3b^2 + b \\ 3c^2 + c \end{bmatrix}$$

$$\Rightarrow 4a^2 f(-1) + 4a f(1) + f(2) = 3a^2 + a \quad \dots(1)$$

$$4b^2 f(-1) + 4b f(1) + f(2) = 3b^2 + b \quad \dots(2)$$

$$4c^2 f(-1) + 4c f(1) + f(2) = 3c^2 + c \quad \dots(3)$$

where  $f(x)$  is quadratic expression given by,

$$f(x) = ax^2 + bx + c \text{ and (1), (2) and (3)}$$

$$\Rightarrow 4x^2 f(-1) + 4x f(1) + f(2) = 3x^2 + 3x$$

$$\text{or } (4f(-1) - 3)x^2 + (4f(1) - 3)x + f(2) = 0 \quad \dots(4)$$

As above equation has 3 roots  $a$ ,  $b$  and  $c$ .

$\therefore$  above equation is identity in  $x$ .

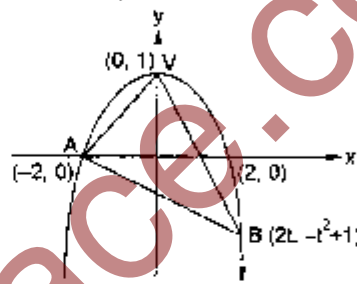
i.e., Coefficients must be zero

$$\Rightarrow f(-1) = 3/4, f(1) = 3/4, f(2) = 0 \quad \dots(5)$$

$$\therefore f(x) = ax^2 + bx + c$$

$$\Rightarrow a = -1/4, b = 0 \text{ and } c = 1, \text{ using (5)}$$

Thus,  $f(x) = \frac{4-x^2}{4}$  shown as,



$$\text{Let } A(-2, 0), B = (2t, -t^2 + 1)$$

Since  $AB$  subtends right angle at vertex  $V(0, 1)$

$$\Rightarrow \frac{1}{2} \cdot \frac{-t^2}{2t} = -1$$

$$\Rightarrow t = 4$$

$$\therefore B(8, -15)$$

$$\text{Thus, Area} = \int_{-2}^8 \left( \frac{4-x^2}{4} + \frac{3x+6}{2} \right) dx$$

$$= \left[ x - \frac{x^3}{12} + \frac{3x^2}{4} + 3x \right]_{-2}^8$$

$$= \frac{125}{3} \text{ square units}$$

□