

# Limits

## Definitions

**Precise Definition :** We say  $\lim_{x \rightarrow a} f(x) = L$  if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that whenever  $0 < |x - a| < \delta$  then  $|f(x) - L| < \varepsilon$ .

**“Working” Definition :** We say  $\lim_{x \rightarrow a} f(x) = L$  if we can make  $f(x)$  as close to  $L$  as we want by taking  $x$  sufficiently close to  $a$  (on either side of  $a$ ) without letting  $x = a$ .

**Right hand limit :**  $\lim_{x \rightarrow a^+} f(x) = L$ . This has the same definition as the limit except it requires  $x > a$ .

**Left hand limit :**  $\lim_{x \rightarrow a^-} f(x) = L$ . This has the same definition as the limit except it requires  $x < a$ .

**Limit at Infinity :** We say  $\lim_{x \rightarrow \infty} f(x) = L$  if we can make  $f(x)$  as close to  $L$  as we want by taking  $x$  large enough and positive.

There is a similar definition for  $\lim_{x \rightarrow -\infty} f(x) = L$  except we require  $x$  large and negative.

**Infinite Limit :** We say  $\lim_{x \rightarrow a} f(x) = \infty$  if we can make  $f(x)$  arbitrarily large (and positive) by taking  $x$  sufficiently close to  $a$  (on either side of  $a$ ) without letting  $x = a$ .

There is a similar definition for  $\lim_{x \rightarrow a} f(x) = -\infty$  except we make  $f(x)$  arbitrarily large and negative.

### Relationship between the limit and one-sided limits

$$\lim_{x \rightarrow a} f(x) = L \Rightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L \quad \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^-} f(x) = L \Rightarrow \lim_{x \rightarrow a} f(x) = L$$

$$\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x) \Rightarrow \lim_{x \rightarrow a} f(x) \text{ Does Not Exist}$$

### Properties

Assume  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  both exist and  $c$  is any number then,

- $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  provided  $\lim_{x \rightarrow a} g(x) \neq 0$
- $\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n$
- $\lim_{x \rightarrow a} \left[ \sqrt[n]{f(x)} \right] = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

### Basic Limit Evaluations at $\pm \infty$

Note  $\operatorname{sgn}(a) = 1$  if  $a > 0$  and  $\operatorname{sgn}(a) = -1$  if  $a < 0$ .

- $\lim_{x \rightarrow \infty} e^x = \infty$  &  $\lim_{x \rightarrow -\infty} e^x = 0$
- $\lim_{x \rightarrow \infty} \ln(x) = \infty$  &  $\lim_{x \rightarrow 0^-} \ln(x) = -\infty$
- If  $r > 0$  then  $\lim_{x \rightarrow \infty} \frac{b}{x^r} = 0$
- If  $r > 0$  and  $x^r$  is real for negative  $x$  then  $\lim_{x \rightarrow -\infty} \frac{b}{x^r} = 0$
- $n$  even :  $\lim_{x \rightarrow \pm \infty} x^n = \infty$
- $n$  odd :  $\lim_{x \rightarrow \infty} x^n = \infty$  &  $\lim_{x \rightarrow -\infty} x^n = -\infty$
- $n$  even :  $\lim_{x \rightarrow \pm \infty} ax^n + \dots + bx + c = \operatorname{sgn}(a)\infty$
- $n$  odd :  $\lim_{x \rightarrow \infty} ax^n + \dots + bx + c = \operatorname{sgn}(a)\infty$
- $n$  odd :  $\lim_{x \rightarrow -\infty} ax^n + \dots + bx + c = -\operatorname{sgn}(a)\infty$

## Evaluation Techniques

### Continuous Functions

If  $f(x)$  is continuous at  $a$  then  $\lim_{x \rightarrow a} f(x) = f(a)$

### Continuous Functions and Composition

$f(x)$  is continuous at  $b$  and  $\lim_{x \rightarrow a} g(x) = b$  then

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(b)$$

### Factor and Cancel

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{x^2 - 2x} &= \lim_{x \rightarrow 2} \frac{(x-2)(x+6)}{x(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{x+6}{x} = \frac{8}{2} = 4 \end{aligned}$$

### Rationalize Numerator/Denominator

$$\begin{aligned} \lim_{x \rightarrow 9} \frac{3 - \sqrt{x}}{x^2 - 81} &= \lim_{x \rightarrow 9} \frac{3 - \sqrt{x}}{x^2 - 81} \cdot \frac{3 + \sqrt{x}}{3 + \sqrt{x}} \\ &= \lim_{x \rightarrow 9} \frac{9 - x}{(x^2 - 81)(3 + \sqrt{x})} = \lim_{x \rightarrow 9} \frac{-1}{(x+9)(3 + \sqrt{x})} \\ &= \frac{-1}{(18)(6)} = -\frac{1}{108} \end{aligned}$$

### Combine Rational Expressions

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{1}{x+h} - \frac{1}{x} \right) &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{x - (x+h)}{x(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{-h}{x(x+h)} \right) = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2} \end{aligned}$$

### Some Continuous Functions

Partial list of continuous functions and the values of  $x$  for which they are continuous.

- Polynomials for all  $x$ .
- Rational function, except for  $x$ 's that give division by zero.
- $\sqrt[n]{x}$  ( $n$  odd) for all  $x$ .
- $\sqrt{x}$  ( $n$  even) for all  $x \geq 0$ .
- $e^x$  for all  $x$ .
- $\ln x$  for  $x > 0$ .
- $\cos(x)$  and  $\sin(x)$  for all  $x$ .
- $\tan(x)$  and  $\sec(x)$  provided  $x \neq \dots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$
- $\cot(x)$  and  $\csc(x)$  provided  $x \neq \dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots$

### Intermediate Value Theorem

Suppose that  $f(x)$  is continuous on  $[a, b]$  and let  $M$  be any number between  $f(a)$  and  $f(b)$ .

Then there exists a number  $c$  such that  $a < c < b$  and  $f(c) = M$ .

### L'Hospital's Rule

If  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$  or  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\pm \infty}{\pm \infty}$  then,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \quad a \text{ is a number, } \infty \text{ or } -\infty$$

### Polynomials at Infinity

$p(x)$  and  $q(x)$  are polynomials. To compute

$\lim_{x \rightarrow \pm \infty} \frac{p(x)}{q(x)}$  factor largest power of  $x$  out of both

$p(x)$  and  $q(x)$  and then compute limit.

$$\lim_{x \rightarrow -\infty} \frac{3x^2 - 4}{5x - 2x^2} = \lim_{x \rightarrow -\infty} \frac{x^2 \left( 3 - \frac{4}{x^2} \right)}{x^2 \left( \frac{5}{x} - 2 \right)} = \lim_{x \rightarrow -\infty} \frac{3 - \frac{4}{x^2}}{\frac{5}{x} - 2} = -\frac{3}{2}$$

### Piecewise Function

$$\lim_{x \rightarrow -2} g(x) \text{ where } g(x) = \begin{cases} x^2 + 5 & \text{if } x < -2 \\ 1 - 3x & \text{if } x \geq -2 \end{cases}$$

Compute two one sided limits,

$$\lim_{x \rightarrow -2^-} g(x) = \lim_{x \rightarrow -2^-} x^2 + 5 = 9$$

$$\lim_{x \rightarrow -2^+} g(x) = \lim_{x \rightarrow -2^+} 1 - 3x = 7$$

One sided limits are different so  $\lim_{x \rightarrow -2} g(x)$

doesn't exist. If the two one sided limits had been equal then  $\lim_{x \rightarrow -2} g(x)$  would have existed and had the same value.