6. Some Simple Transformations of Quadratic Equations.

Let $ax^2 + bx + c = 0$ ....(1)

be a quadratic equation. Without solving this equation, we can often change it into another equation whose roots are related to the roots of (1) in some manner or other. This process is known as a **Transformation** of (1).

Working Rules: Let α, β be the roots of (1).

(*i*) **Roots with signs charged:** The quadratic equation whose roots are $(-\alpha)$ and $(-\beta)$ is obtained by replacing x in (1) by $(-x)$.

(*ii*) **Reciprocal roots:** The quadratic equation whose roots are $(1/\alpha)$ and $(1/\beta)$ is obtained by replacing x in (1) by $(1/x)$.

(*iii*) **Roots multiplied by a given constant k:** The quadratic equation whose roots are $(k\alpha)$ and $(k\beta)$ is obtained by multiplying the successive coefficients in (1) by 1, k, k^2 respectively.

1. Arithmetic Progression:

(*i*) The *nth* term: $t_n = a + (n - 1)\, d$.

(*ii*) The sum of n terms: $S_n = \frac{n}{2}\,[a + l]$ or,

$$S_n = \left(\frac{n}{2}\right)[2a + (n-1)\,d]$$

(*iii*) *Three* numbers in *A.P.*: $a - d$, a, $a + d$.

(*iv*) *Five* numbers in *A.P.*: $a - 2d$, $a - d$, a, $a + d$, $a + 2d$.

(*v*) *Four* numbers in *A.P.*: $a - 3d$, $a - d$, $a + d$, $a + 3d$.

(*vi*) Arithmetic mean of a and b is $A = \frac{(a+b)}{2}$.

2. Geometric Progression:

(*i*) The *nth* term: $t_n = ar^{n-1}$

(*ii*) The sum of n terms: $S_n = \frac{a(1-r^n)}{(1-r)}$, or,

$$S_n = \frac{a - rt_n}{(1-r)}.$$

(*iii*) *Three* numbers & in G.P.: a/r, a, ar.

(*iv*) *Four* numbers & in G.P.: a/r^3, a/r, ar, ar^3.

(*v*) Sum of infinite no. of terms: $S_\infty = \frac{a}{(1-r)}$,

provided $|r| < 1$.

(*vi*) Geometric mean of a and b: $G = \sqrt{ab}$

3. Harmonic Progression:

(*i*) The *nth* term: $t_n = \frac{1}{a + (n-1)\,d}$.

(*ii*) Harmonic mean of a and b: $H = \frac{2ab}{a+b}$

4. Relation among Various means:

(*i*) A, G and H are in G.P., i.e., $G^2 = AH$.

(*ii*) $A \geq G \geq H$.

5. Some well-known Sums:

(*i*) $\Sigma n = n(n+1)/2$.

(*ii*) $\Sigma n^2 = n(n+1)(2n+1)/6$.

(*iii*) $\Sigma n^3 = (\Sigma n)^2 = [n(n+1)/2]^2$.

6. Miscellaneous Series:

(*i*) $S_n = \Sigma t_n$

(*ii*) $S_\infty = \lim_{n\to\infty} S_n$

PERMUTATIONS:

(*i*) The no. of permutations of n different objects taken r at a time is given by ${}^nP_r = \frac{n!}{(n-r)!}$.

(*ii*) If P denotes the number of permutations of n things taken all at a time where p are of one kind, q are of another kind, and r are of third kind, and so on, and the rest are all different,

then $P = \frac{n!}{p!\,q!\,r!\,...}$.

(*iii*) If *repetition is allowed*, the no. of permutation of n distinct objects taken r at a time is $P = n^r$.

(*iv*) The no. of *circular* permutations of n dissimilar things taken *all* at a time is $P = (n-1)!$

COMBINATIONS

(i) The no. of combinations of n different objects taken r at a time is given by

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

(ii) $^nP_r = r!\,^nC_r$

(iii) $^nC_r = {}^nC_{n-r}$

(iv) $^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$ (*Pascal's rule*)

(v) $^nC_r / {}^{n-1}C_{r-1} = n/r$.

(vi) $^nC_{r+1} = \left(\frac{n-r}{r+1}\right) {}^nC_r,\ 0 \le r \le n$.

(vii) The no. of diagonals in an *n-sided* closed polygons $= {}^nC_2 - n$.

The Binomial Theorem:

(i) $(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_n a^n$, *where* $n \in N$.

(ii) The $(r+1)$*th term*: $T_{r+1} = {}^nC_r x^{n-r} a^r$.

(iii) If n is an *even* natural no., then $\left(\frac{n+2}{2}\right)$th term is the middle term. If n is an *odd* natural no. there are two middle terms namely $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms.

(iv) **For a natural no.** n,
$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$, where $T_{r+1} = {}^nc_r x^r$.

(v) $c_0 + c_1 + c_2 + \ldots + c_n = 2^n$.

(vi) $c_0 + c_2 + c_4 + \ldots = c_1 + c_3 + c_5 + \ldots = 2^{n-1}$.

(vii) $c_0 - c_1 + c_2 - c_3 + \ldots + (-1)^n c_n = 0$.

(viii) *If n is a negative integer or fractional and* $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots \textit{ ad inf.}$$

Here $T_{r+1} = \frac{n(n-1)\ldots(n-r+1)}{r!}x^r$.

(ix) If n is a *negative integer* or fractional, and $|x| < 1$ then the Binomial expansion of $(1+x)^n$ contains an infinite no. of terms.

(x) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$ ad inf., where $|x| < 1$.

(xi) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$ ad inf., where $|x| < 1$.

(xii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$ ad inf., where $|x| < 1$.

(xiii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$ ad inf., where $|x| < 1$.

(xiv) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots$ ad inf., where $|x| < 1$.

(xv) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots$ ad inf., where $|x| < 1$.

Logrithem and Exponential Series:

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ ad. inf

(ii) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$ ad. inf.

(iii) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$ ad. inf.

(iv) $a^x = 1 + x\,(\log a) + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 \ldots$ ad. inf.

(v) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ ad. inf.

(vi) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$ ad. inf.

Trigonometry:

Sums of angles:

(i) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

(ii) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

(iii) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

(iv) $\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$.

(v) $\tan(\alpha+\beta+\gamma)$.

$$= \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-(\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha)}$$

Difference of angles:

(i) $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$.

(ii) $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

(iii) $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$.

(iv) $\cot(\alpha-\beta)=\dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$.

Multiple angles:

(i) $\sin 2\theta=2\sin\theta\cos\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}$,

(ii) $\cos 2\theta=\cos^2\theta-\sin^2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

In particular, $1+\cos 2\theta=2\cos^2\theta$

and $1-2\theta=2\sin^2\theta$.

(iii) $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

(iv) $\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$.

(v) $\sin 3\theta=3\sin\theta-4\sin^3\theta$.

(vi) $\cos 3\theta=4\cos^3\theta-3\cos\theta$.

Tansformations of Products into Sums or Differences:

(i) $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$.

(ii) $2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$

(iii) $2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$.

(iv) $2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$.

Transformations of sum or a difference into Products:

(i) $\sin C+\sin D=2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$.

(ii) $\sin C-\sin D=2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$.

(iii) $\cos C+\cos D=2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$.

(iv) $\cos C-\cos D=2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

Half-angle Formulas:

(i) $\sin\left(\dfrac{A}{2}\right)=\pm\sqrt{\left(\dfrac{1-\cos A}{2}\right)}$

Trigonometrical Equations:

(i) **Sine Equation:** If $\sin\theta=\sin\alpha$, then $\theta=n\pi+(-1)^n\alpha$, $n\in Z$.

(ii) **Cosine Equation:** If $\cos\theta=\cos\alpha$, then $\theta=2n\pi\pm\alpha$, $n\in Z$.

(iii) **Tangent Equation:** If $\tan\theta=\tan\alpha$, then $\theta=n\pi+\alpha$, $n\in Z$.

Note that θ and α are measured in radians.

Properties of Triangles:

(i) **Sine Formula:** $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$.

(ii) **Cosine Formula:** $\cos A=(b^2+c^2-a^2)/2bc$.

$\cos B=(c^2+a^2-b^2)/2ca$.

$\cos C=(a^2+b^2-c^2)/2ab$.

Napier's Analogies:

(i) $\tan\left(\dfrac{A-B}{2}\right)=\dfrac{a-b}{a+b}\cot\left(\dfrac{C}{2}\right)$.

(ii) $\tan\left(\dfrac{B-C}{2}\right)=\dfrac{b-c}{b+c}\cot\left(\dfrac{A}{2}\right)$.

(iii) $\tan\left(\dfrac{C-A}{2}\right)=\dfrac{c-a}{c+a}\cot\left(\dfrac{B}{2}\right)$

Area of a triangle:

(i) $\Delta=½\,ab\sin C=½\,bc\sin A=½\,ca\sin B$

(ii) $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$,

where $s=½(a+b+c)$

Sines of half angles in terms of sides:

(*i*) $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$,

(*ii*) $\sin\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{ca}}$,

(*iii*) $\sin\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{bc}}$.

Cosines of half angles in terms of sides:

(*i*) $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}$,

(*ii*) $\cos\left(\frac{B}{2}\right) = \sqrt{\frac{s(s-b)}{ca}}$,

(*iii*) $\cos\left(\frac{C}{2}\right) = \sqrt{\frac{s(s-c)}{ab}}$.

Tangents of half angles in terms of sides:

(*i*) $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$,

(*ii*) $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$,

(*iii*) $\tan\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$.

Radii off Circumcircle and Incircle:

(*i*) $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

(*ii*) $R = \frac{abc}{4\Delta}$,

(iii) $r = \frac{\Delta}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$,

Regular Polygon:

(*i*) The circum-radius of a regular polygon of *n* sides, each equal to '*a*' is

$R = \frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{n}\right)$.

(*ii*) The in-radius of a regular polygon of *n* sides, each equal to '*a*' is

$r = \frac{a}{2}\cot\left(\frac{\pi}{n}\right)$,

(*iii*) Area A of a regular polygon of *n* sides is given by

(*a*) $A = \frac{na^2}{4}\cot\left(\frac{\pi}{n}\right)$ (In terms of side),

(*b*) $A = \frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$ (In terms of circum-radius).

(*c*) $A = nr^2\tan\left(\frac{\pi}{n}\right)$ (In terms of in-radius).

Inverse Function:

1. $\sin^{-1}(\sin\theta) = \theta$; $\cos^{-1}(\cos\theta) = \theta$;
$\tan^{-1}(\tan\theta) = \theta$; $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$;
$\sec^{-1}(\sec\theta) = \theta$; $\cot^{-1}(\cot\theta) = \theta$

2. $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1}x$; $\operatorname{cosec}^{-1} = \sin^{-1}\left(\frac{1}{x}\right)$.

3. $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}x$; $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$.

4. $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$; $\tan^{-1}(x) = \cot^{-1}\left(\frac{1}{x}\right)$.

5. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$; $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$.

6. $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$.

7. $\sin^{-1}(-x) = -\sin^{-1}x$;
$\tan^{-1}(-x) = -\tan^{-1}x$,
$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$.

8. $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $xy<1$

9. $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$.

10. $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), -1 < x < 1.$

11. $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right), -1 \le x \le 1,$

12. $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 \le x < \infty$

MATRIX

1. A rectangular arrangement of numbers (real or complex) is called an $m \times n$ matrix and is written as $A = (a_{ij})$, i = 1, 2, ..., n. The quantities a_{ij} are called the **elements** of the matrix. If $m = n$, A is said to be a **square matrix**.

2. (*i*) If a matrix has only one row, it is called a **row matrix**.
   (*ii*) If a matrix has only one column, it is called a **column matrix**.
   (*iii*) A *square* matrix I in which all the elements of the principal digonal equal to 1 and the other elements are zeros, is called an **Identity matrix**.
   (*iv*) A matrix O in which all the elements are zeros, is called the **null matrix**.
   (*v*) For complex numbers a_{ij}, the matrix $\bar{A} = (\bar{a}_{ij})$ is called the conjugate matrix of A.

3. If $A = (a_{ij})$ and $B = (b_{ij})$ have the same order, we say that $A = B \Leftrightarrow a_{ij} = b_{ij}$.

4. Two matrices may be added or subtracted if they have the same order. The sum of two matrices is obtained by adding the corresponding elements of the two matrices.

5. If λ is a real number and $A = (a_{ij})$ is any matrix, then $\lambda A = (\lambda a_{ij})$. In particular, if λ = –1 then $-A = (-a_{ij})$.

6. Let A, B, C be matrices of the same order and λ, μ are scalars, then the following laws are satisfied:
   (*i*) $A + B = B + A$.
   (*ii*) $A + (B + C) = (A + B) + C$
   (*iii*) $A + 0 = 0 + A = A$.
   (*iv*) $A + (-A) = (-A) + A = 0$.
   (*v*) $A + B = A + C \Rightarrow B = C$.
   (*vi*) $\lambda(A + B) = \lambda A + \lambda B$.
   (*vii*) $(\lambda + \mu) A = \lambda A + \mu A$.
   (*viii*) $\lambda(\mu A) = (\lambda\mu) A$.

7. (*i*) If $A = (a_{ij})$ is a matrix of order $m \times n$ then its **transpose** $A' = (a_{ji})$ is a matrix of order $n \times m$.
   (*ii*) $(A')' = A$ and $(A + B)' = A' + B'$.
   (*iii*) A square matrix $A = (a_{ij})$ is said to be *symmetric* if $A' = A$, and skew symmetric if $A' = -A$.

8. (*i*) For two given matrices A and B, AB exists if the number of *columns* in A equal to number of *rows* in B. Thus if $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{jk})$ is an $n \times p$ matrix, then AB exists and is a matrix of order $m \times p$, given by
   $AB = (C_{ik})$,
   where C_{ik} is obtained by multiplying each element of the *i*th *row* of A by the corresponding elements in the *k*th *column* of B and then adding them.
   (*ii*) $AA = A^2$, $AAA = A^3$ *etc.*
   (*iii*) In general, $(A + B)^2 \ne A^2 + 2AB + B^2$.

9. If A, B and C are any three matrices such that the stated operations may be performed, then the following properties are satisfied:
   (*i*) $A(BC) = (AB)C$.
   (*ii*) $A(B + C) = AB + AC$, and $(A + B)C = AC + BC$.
   (*iii*) In general, $AB \ne BA$.
   (*iv*) The product of two non-null matrices may be a null matrix.
   (*v*) Cancellation law w.r.t. multiplication does not hold.
   (*vi*) $(AB)' = B'A'$ (Reversal law for transpose of product).