Solution to INMO-2002 Problems

1. For a convex hexagon $ABCDEF$ in which each pair of opposite sides is unequal, consider the following six statements:

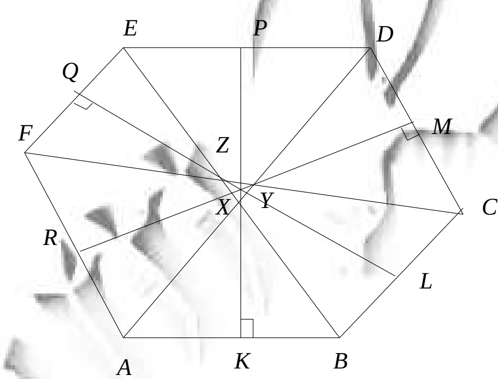
   (a_1) AB is parallel to DE; (a_2) $AE = BD$;
   (b_1) BC is parallel to EF; (b_2) $BF = CE$;
   (c_1) CD is parallel to FA; (c_2) $CA = DF$.

   (a) Show that if all the six statements are true, then the hexagon is cyclic(i.e., it can be inscribed in a circle).

   (b) Prove that, in fact, any five of these six statements also imply that the hexagon is cyclic.

Solution:

(a) Suppose all the six statements are true. Then $ABDE$, $BCEF$, $CDFA$ are isosceles trapeziums; if K, L, M, P, Q, R are the mid-points of AB, BC, CD, DE, EF, FA respectively, then we see that $KP \perp AB, ED$; $LQ \perp BC, EF$ and $MR \perp CD, FA$.



If AD, BE, CF themselves concur at a point O, then $OA = OB = OC = OD = OE = OF$. ($O$ is on the perpendicular bisector of each of the sides.) Hence A, B, C, D, E, F are concyclic and lie on a circle with centre O. Otherwise these lines AD, BE, CF form a triangle, say XYZ. (See Fig.) Then KX, MY, QZ, when extended, become the internal angle bisectors of the triangle XYZ and hence concur at the incentre O' of XYZ. As earlier O' lies on the perpendicular bisector of each of the sides. Hence $O'A = O'B = O'C = O'D = O'E = O'F$, giving the concyclicity of A, B, C, D, E, F.

(b) Suppose (a_1), (a_2), (b_1), (b_2) are true. Then we see that $AD = BE = CF$. Assume that (c_1) is true. Then CD is parallel to AF. It follows that triangles YCD and YFA are similar. This gives

$$\frac{FY}{AY} = \frac{YC}{YD} = \frac{FY + YC}{AY + YD} = \frac{FC}{AD} = 1.$$

We obtain $FY = AY$ and $YC = YD$. This forces that triangles CYA and DYF are congruent. In particular $AC = DF$ so that (c_2) is true. The conclusion follows from (a). Now assume that (c_2) is true; i.e., $AC = FD$. We have seen that $AD = BE = CF$. It follows that triangles FDC and ACD are congruent. In particular $\angle ADC = \angle FCD$. Similarly, we can show that $\angle CFA = \angle DAF$. We conclude that CD is parallel to AF giving (c_1).

2. Determine the least positive value taken by the expression $a^3+b^3+c^3-3abc$ as a, b, c vary over all positive integers. Find also all triples (a, b, c) for which this least value is attained.

Solution: We observe that

$$Q = a^3 + b^3 + c^3 - 3abc = \frac{1}{2}\Big(a+b+c\Big)\Big((a-b)^2 + (b-c)^2 + (c-a)^2\Big).$$

Since we are looking for the least positive value taken by Q, it follows that a, b, c are not all equal. Thus $a + b + c \geq 1 + 1 + 2 = 4$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 1 + 1 + 0 = 2$. Thus we see that $Q \geq 4$. Taking $a = 1$, $b = 1$ and $c = 2$, we get $Q = 4$. Therefore the least value of Q is 4 and this is achieved only by $a+b+c = 4$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 = 2$. The triples for which $Q = 4$ are therefore given by

$$(a, b, c) = (1, 1, 2),\ (1, 2, 1),\ (2, 1, 1).$$

3. Let x, y be positive reals such that $x + y = 2$. Prove that

$$x^3y^3(x^3 + y^3) \leq 2.$$

Solution: We have from the AM-GM inequality, that

$$xy \leq \left(\frac{x+y}{2}\right)^2 = 1.$$

Thus we obtain $0 < xy \leq 1$. We write

$$\begin{aligned} x^3y^3\left(x^3+y^3\right) &= \left(xy\right)^3\left(x+y\right)\left(x^2-xy+y^2\right) \\ &= 2\left(xy\right)^3\Big(\left(x+y\right)^2 - 3xy\Big) \\ &= 2\left(xy\right)^3\left(4-3xy\right). \end{aligned}$$

Thus we need to prove that

$$(xy)^3(4-3xy) \leq 1.$$

Putting $z = xy$, this inequality reduces to

$$z^3(4-3z) \leq 1,$$

for $0 < z \leq 1$. We can prove this in different ways. We can put the inequality in the form

$$3z^4 - 4z^3 + 1 \geq 0.$$

Here the expression in the **LHS** factors to $(z-1)^2(3z^2+2z+1)$ and $(3z^2+2z+1)$ is positive since its discriminant $D = -8 < 0$. Or applying the AM-GM inequality to the positive reals $4-3z, z, z, z$, we obtain

$$z^3(4-3z) \leq \left(\frac{4-3z+3z}{4}\right)^4 \leq 1.$$

4. Do there exist 100 lines in the plane, no three of them concurrent, such that they intersect exactly in 2002 points?

Solution: Any set of 100 lines in the plane can be partitioned into a finite number of disjoint sets, say $A_1, A_2, A_3, \ldots, A_k$, such that

(i) Any two lines in each A_j are parallel to each other, for $1 \leq j \leq k$ (provided, of course, $|A_j| \geq 2$);

(ii) for $j \neq l$, the lines in A_j and A_l are not parallel.

If $|A_j| = m_j$, $1 \leq j \leq k$, then the total number of points of intersection is given by $\sum_{1 \leq j \leq l \leq k} m_j m_l$, as no three lines are concurrent. Thus we have to find positive integers $m_1, m_2, \ldots, m_k$ such that

$$\sum_{j=1}^{k} m_j = 100, \quad \sum m_j m_l = 2002,$$

for an affirmative answer to the given question.

We observe that

$$\begin{aligned} \sum_{j=1}^{k} m_j^2 &= \left(\sum_{j=1}^{k} m_j\right)^2 - 2\left(\sum m_j m_l\right) \\ &= 100^2 - 2(2002) = 5996. \end{aligned}$$

Thus we have to choose $m_1, m_2, \dots, m_k$ such that

$$\sum_{j=1}^{k} m_j = 100, \quad \sum_{j=1}^{k} m_j^2 = 5996.$$

We observe that $\left[\sqrt{5996}\ \right] = 77$. So we may take $m_1 = 77$, so that

$$\sum_{j=2}^{k} m_j = 23, \quad \sum j = 2^k m_j^2 = 67.$$

Now we may choose $m_2 = 5$, $m_3 = m_4 = 4$, $m_5 = m_6 = \cdots = m_{14} = 1$. Finally, we can take

$$k = 14, \quad (m_1, m_2, \dots, m_{14}) = (77, 5, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

proving the existence of 100 lines with exactly 2002 points of intersection.

5. Do there exist three distinct positive real numbers a, b, c such that the numbers a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order?

Solution: We show that the answer is **NO**. Suppose, if possible, let a, b, c be three distinct positive real numbers such that a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order. We may assume that $a < b < c$. Then there are only two cases we need to check: (I) $a+b-c < a < c+a-b < b < c < b+c-a < a+b+c$ and (II) $a+b-c < a < b < c+a-b < c < b+c-a < a+b+c$.

Case I. Suppose the chain of inequalities $a+b-c < a < c+a-b < b < c < b+c-a < a+b+c$ holds good. let d be the common difference. Thus we see that

$$c = a+b+c-2d,\ b = a+b+c-3d,\ a = a+b+c-5d.$$

Adding these, we see that $a+b+c = 5d$. But then $a = 0$ contradicting the positivity of a.

Case II. Suppose the inequalities $a+b-c < a < b < c+a-b < c < b+c-a < a+b+c$ are true. Again we see that

$$c = a+b+c-2d,\ b = a+b+c-4d,\ a = a+b+c-5d.$$

We thus obtain $a+b+c = (11/2)d$. This gives

$$a = \frac{1}{2}d,\ b = \frac{3}{2}d,\ c = \frac{7}{2}d.$$

Note that $a+b-c = a+b+c-6d = -(1/2)d$. However we also get $a+b-c = \big[(1/2)+(3/2)-(7/2)\big]d = -(3/2)d$. It follows that $3e = e$ giving $d = 0$. But this is impossible.

Thus there are no three distinct positive real numbers a, b, c such that a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order.

6. Suppose the n^2 numbers $1, 2, 3, \ldots, n^2$ are arranged to form an n by n array consisting of n rows and n columns such that the numbers in each row(from left to right) and each column(from top to bottom) are in increasing order. Denote by a_{jk} the number in j-th row and k-th column. Suppose b_j is the maximum possible number of entries that can occur as a_{jj}, $1 \le j \le n$. Prove that

$$b_1 + b_2 + b_3 + \cdots b_n \le \frac{n}{3}\big(n^2 - 3n + 5\big).$$

(Example: In the case $n = 3$, the only numbers which can occur as a_{22} are 4, 5 or 6 so that $b_2 = 3$.)

Solution: Since a_{jj} has to exceed all the numbers in the top left $j \times j$ submatrix (excluding itself), and since there are $j^2 - 1$ entries, we must have $a_{jj} \ge j^2$. Similarly, a_{jj} must not exceed eac of the numbers in the bottom right $(n-j+1) \times (n-j+1)$ submatrix (other than itself) and there are $(n-j+1)^2 - 1$ such entries giving $a_{jj} \le n^2 - (n-j+1)^2 + 1$. Thus we see that

$$a_{jj} \in \Big\{j^2, j^2+1, j^2+2, \ldots, n^2 - (n-j+1)^2 + 1\Big\}.$$

The number of elements in this set is $n^2 - (n-j+1)^2 - j^2 + 2$. This implies that

$$b_j \le n^2 - (n-j+1)^2 - j^2 + 2 = (2n+2)j - 2j^2 - (2n-1).$$

It follows that

$$\begin{aligned}
\sum_{j=1}^{n} b_j &\le \Big(2n+2\Big)\sum_{j=1}^{n} j - 2\sum_{j=1}^{n} j^2 - n(2n-1) \\
&= \Big(2n+2\Big)\left(\frac{n(n+1)}{2}\right) - 2\left(\frac{n(n+1)(2n+1)}{6}\right) - n(2n-1) \\
&= \frac{n}{3}\Big(n^2 - 3n + 5\Big),
\end{aligned}$$

which is the required bound.