A FILL IN THE BLANKS

1. The sum of the coefficients of the polynomial $(1+x-3x^2)^{2163}$ is .... **(IIT 1982)**

2. The coefficient of x^{99} in the polynomial $(x-1)(x-2)\ldots(x-100)$ is ... **(IIT 1982; 2M)**

3. If $(1+ax)^n = 1+8x+24x^2+\ldots$, then $a = \ldots$ and $n = \ldots$ **(IIT 1983; 2M)**

4. Let n be a positive integer. If the coefficients of 2nd, 3rd, and 4th terms in the expansion of $(1+x)^n$ are in A.P., then the value of n is... **(IIT 1994; 2M)**

C OBJECTIVE QUESTIONS

➡ *Only one option is correct :*

1. Given positive integers $r>1$, $n>2$ and the coefficient of $(3r)$ *th* and $(r+2)$*th* terms in the binomial expansion of $(1+x)^{2n}$ are equal. Then : **(IIT 1980)**

   (a) $n=2r$ (b) $n=2r+1$
   (c) $n=3r$ (d) none of these

2. The coefficient of x^4 in $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is : **(IIT 1983; 1M)**

   (a) $\frac{405}{256}$ (b) $\frac{504}{259}$
   (c) $\frac{450}{263}$ (d) none of these

3. If C_r stands for ${}^nC_r$, then the sum of the series

   $$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2-2C_1^2+3C_2^2-\ldots+(-1)^n(n+1)C_n^2]$$

   where n is an even positive integer, is equal to : **(IIT 1986; 2M)**

   (a) $(-1)^{n/2}(n+2)$ (b) $(-1)^n(n+1)$
   (c) $(-1)^{n/2}(n+1)$ (d) none of these

4. The expression

   $$\{x+(x^3-1)^{1/2}\}^5+\{x-(x^3-1)^{1/2}\}^5$$

   is a polynomial of degree : **(IIT 1992; 2M)**

   (a) 5 (b) 6
   (c) 7 (d) 8

5. If $a_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n}\frac{r}{{}^nC_r}$ equals : **(IIT 1998; 2M)**

   (a) $(n-1)\,a_n$ (b) $n\,a_n$
   (c) $\frac{1}{2}\,na_n$ (d) none of these

6. If in the expansion of $(1+x)^m(1-x)^n$, the coefficients of x and x^2 are 3 and -6 respectively, then m is : **(IIT 1999; 2M)**

   (a) 6 (b) 9
   (c) 12 (d) 24

7. For $2\le r\le n$, $\binom{n}{r}+2\binom{n}{r-1}+\binom{n}{r-2}$ is equal to : **(IIT 2000)**

   (a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$
   (c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$

8. In the binomial expansion of $(a-b)^n$, $n\ge 5$ the sum of the 5^{th} and 6^{th} terms is zero. Then a/b equals : **(IIT 2001)**

   (a) $\frac{n-5}{6}$ (b) $\frac{n-4}{5}$
   (c) $\frac{5}{n-4}$ (d) $\frac{6}{n-5}$

9. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1}-T_n=21$, then n equals : **(IIT 2001)**

   (a) 5 (b) 7
   (c) 6 (d) 4

10. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, where $\binom{p}{q}=0$ if $p>q$, is maximum when m is : **(IIT 2002)**

    (a) 5 (b) 10
    (c) 15 (d) 20

11. Coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is : **(IIT 2003)**

(a) $^{12}C_6 + 3$ (b) $^{12}C_6 + 1$
(c) $^{12}C_6$ (d) $^{12}C_6 + 2$

12. If $^{n-1}C_r = (k^2 - 3)\,^nC_{r+1}$, then $k \in$: **(IIT 2004)**

(a) $(-\infty, -2]$ (b) $[2, \infty)$
(c) $[-\sqrt{3}, \sqrt{3}]$ (d) $(\sqrt{3}, 2]$

13. $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \dots + \binom{30}{20}\binom{30}{30} =$ **(IIT 2005)**

(a) $^{30}C_{11}$ (b) $^{60}C_{10}$
(c) $^{30}C_{10}$ (d) $^{65}C_{55}$

E SUBJECTIVE QUESTIONS

1. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then show that the sum of the products of the C_i's taken two at a time represented by $\sum\sum_{0 \le i < j \le n} C_iC_j$ is equal to

$$2^{2n-1} - \frac{(2n)!}{2(n!)^2}$$

(IIT 1983; 3M)

2. Given $s_n = 1 + q + q^2 + \dots + q^n$

$$S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \dots + \left(\frac{q+1}{2}\right)^n, q \ne 1$$

Prove that

$$^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \dots + {}^{n+1}C_{n+1} s_n = 2^n S_n$$

(IIT 1984; 4M)

3. Find the sum of the series :

$$\sum_{r=0}^{n} (-1)^r\,^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} \dots \text{up to } m \text{ terms}\right]$$

(IIT 1985; 5M)

4. Prove that

$C_0 - 2^2C_1 + 3^2C_2 - \dots + (-1)^n(n+1)^2C_n = 0, n > 2$

where $C_r = {}^nC_r$. **(IIT 1989; 5M)**

5. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$

and $a_k = 1$ for all $k \ge n$, then show that

$$b_n = {}^{2n-1}C_{n+1}$$

(IIT 1992; 6M)

6. Prove that $\sum_{r=1}^{k} (-3)^{r-1}\,^{3n}C_{2r-1} = 0$, where $k = (3n)/2$ and n is an even positive integer. **(IIT 1993; 5M)**

7. Let n be a positive integer and

$$(1 + x + x^2)^n = a_0 + a_1x + \dots + a_{2n}x^{2n}.$$

Show that $a_0^2 - a_1^2 + \dots + a_{2n}^2 = a_n$. **(IIT 1994; 5M)**

8. Prove that

$$\frac{3!}{2(n+3)} = \sum_{r=0}^{n} (-1)^r \left(\frac{^nC_r}{^{r+3}C_r}\right)$$

(IIT 1997C; 5M)

9. Let n be any positive integer. Prove that :

$$\sum_{k=0}^{m} \frac{\binom{2n-k}{k}}{\binom{2n-k}{n}} \cdot \frac{(2n-4k+1)}{(2n-2k+1)} 2^{n-2k} = \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}} 2^{n-2m}$$

for each non-negative integer

$m \le n \left(\text{Here } \binom{p}{q} = {}^pC_q\right)$ **(IIT 1999; 10M)**

10. For any positive integers m, n (with $n \ge m$),

let $\binom{n}{m} = {}^nC_m$. Prove that

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \dots + \binom{m}{m} = \binom{n+1}{m+1}$$

Hence, or otherwise, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \dots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}$$

(IIT 2000; 6M)

11. Prove that

$$2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \dots (-1)^k\binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$$

(IIT 2003; 2M)

ANSWERS

A Fill in the blanks

1. – 1 **2.** – 5050 **3.** $a = 2, n = 4$ **4.** $n = 7$

C Objective Questions (Only one option)

1. (a) **2.** (a) **3.** (a) **4.** (c) **5.** (c) **6.** (c) **7.** (d)
8. (b) **9.** (b) **10.** (c) **11.** (d) **12.** (d) **13.** (c)

E Subjective Questions

3. $\dfrac{2^{mn}-1}{2^{mn}(2^n-1)}$

SOLUTIONS

A FILL IN THE BLANKS

1. Sum of coefficients is obtained by putting $x = 1$

i.e., $(1 + 1 - 3)^{2163} = -1$

Thus sum of the coefficients of the polynomial

$$(1 + x - 3x^2)^{2163} \text{ is } -1$$

2. The coefficient of x^{99} in

$$(x-1)(x-2)\dots(x-100) \text{ is}$$
$$-(1+2+3+\dots\dots+100)$$
$$= -\frac{100}{2}(1+100) = -50(101)$$
$$= -5050$$

3. Here,

$$(1+ax)^n = 1 + 8x + 24x^2 + \dots..$$

$\Rightarrow \quad 1 + anx + \dfrac{n(n-1)}{2!}a^2x^2 + \dots. = 1 + 8x + 24x^2 + \dots$

$\therefore \quad an = 8$ and $a^2\dfrac{n(n-1)}{2} = 24$

or $\quad 8(8-a) = 48$

$$8 - a = 6$$
$$a = 2$$

thus $\quad a = 2$ and $n = 4$

4. The coefficient of 2nd term in the expansion of $(1+x)^n$ is ${}^nC_1$

The coefficient of 3rd term in the expansion of $(1+x)^n$ is ${}^nC_2$

The coefficient of 4th term in the expansion of $(1+x)^n$ is ${}^nC_3$

Now $2({}^nC_2) = {}^nC_1 + {}^nC_3$ (according to given condition)

$\Rightarrow \quad 2\dfrac{n(n-1)}{1\cdot 2} = n + \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$

$\Rightarrow \quad n - 1 = 1 + \dfrac{(n-1)(n-2)}{6}$

$\Rightarrow \quad n - 1 = 1 + \dfrac{n^2 - 2n - n + 2}{6}$

$\Rightarrow \quad n - 1 = 1 + \dfrac{n^2 - 3n + 2}{6}$

$\Rightarrow \quad n - 1 = \dfrac{6 + n^2 - 3n + 2}{6}$

$\Rightarrow \quad 6n - 6 = 6 + n^2 - 3n + 2$

$\Rightarrow \quad n^2 - 9n + 14 = 0$

$\Rightarrow \quad n^2 - 7n - 2n + 14 = 0$

$\Rightarrow \quad n(n-7) - 2(n-7) = 0$

$\Rightarrow \quad (n-2)(n-7) = 0$

$\Rightarrow \quad n = 2$ or $n = 7$

but ${}^nC_3$ is true for $n \geq 3$ therefore $n = 7$ is the answer.

C OBJECTIVE (ONLY ONE OPTION)

1. Here $t_{3r} = {}^{2n}C_{3r-1}(x)^{3r-1}$

and $\quad t_{r+2} = {}^{2n}C_{r+1}(x)^{r+1}$

Given, binomial coefficients of t_{3r} and t_{r+2} are equal

$\Rightarrow \quad {}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\Rightarrow \quad 3r - 1 = r + 1$ or $2n = (3r-1) + (r+1)$

$\Rightarrow \quad 2r = 2$ or $2n = 4r$

$\Rightarrow \quad r = 1$ or $n = 2r$

But $\quad r > 1, \quad \therefore \quad n = 2r$

2. The general term in $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is

$$t_{r+1}=(-1)^r\ {}^{10}C_r\left(\frac{x}{2}\right)^{10-r}\left(\frac{3}{x^2}\right)^r$$

$$=(-1)^r\ {}^{10}C_r\cdot\frac{3^r}{2^{10-r}}\cdot x^{10-3r}$$

for coefficient of x^4, we have $10-3r=4$

$$\Rightarrow \qquad 3r=6 \quad\text{or}\quad r=2$$

$$\therefore \text{Coefficient of } x^4 \text{ in } \left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}=(-1)^2\cdot{}^{10}C_2\cdot\frac{3^2}{2^8}$$

$$=\frac{9\times 45}{256}=\frac{405}{256}$$

3. (a) We have

$$C_0^2-2C_1^2+3C_2^2-4C_3^2+\ldots\ldots+(-1)^n(n+1)C_n^2$$

$$=\{C_0^2-C_1^2+C_2^2-C_3^2\ldots\ldots+(-1)^nC_n^2\}$$

$$-\{C_1^2-2C_2^2+3C_3^2-\ldots\ldots+(-1)^n nC_n^2\}$$

$$=(-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}-(-1)^{\frac{n}{2}-1}\frac{n}{2}\frac{n!}{(n/2)!(n/2)!}$$

$$=(-1)^{n/2}\frac{n!}{(n/2)!(n/2)!}\left(1-\frac{n}{2}\right)$$

$$\therefore \frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}\{C_0^2-2C_1^2+3C_2^2-\ldots\ldots+(-1)^r(n+1)C_n\}^2$$

$$=\frac{2(n/2)!(n/2)!}{n!}(-1)^{n/2}\frac{n!}{(n/2)!(n/2)!}\frac{(n+2)}{2}$$

$$=(-1)^{n/2}(n+2)$$

4. We know that

$$(a+b)^5+(a-b)^5={}^5C_0a^5+{}^5C_1a^4b+{}^5C_2a^3b^2+{}^5C_3a^2b^3+{}^5C_4ab^4+{}^5C_5b^5+{}^5C_0a^5-{}^5C_1a^4b+{}^5C_2a^3b^2-{}^5C_3a^2b^3+{}^5C_4ab^4-{}^5C_5b^5$$

$$=2[a^5+10a^3b^2+10ab^4]$$

$$\Rightarrow\quad [x+(x^3-1)^{1/2}]^5+[x-(x^3-1)^{1/2}]^5$$

$$=2[x^5+10x^3(x^3-1)+10x(x^3-1)^2]$$

Therefore, the given expression is a polynomial of degree 7.

Hence, (c) is the answer.

5. Let $b=\sum_{r=0}^{n}\frac{r}{{}^nC_r}=\sum_{r=0}^{n}\frac{n-(n-r)}{{}^nC_r}$

$$=n\sum_{r=0}^{n}\frac{1}{{}^nC_r}-\sum_{r=0}^{n}\frac{n-r}{{}^nC_r}$$

$$=na_n-\sum_{r=0}^{n}\frac{n-r}{{}^nC_{n-r}}\qquad[\because {}^nC_r={}^nC_{n-r}]$$

$$=na_n-b$$

$$\Rightarrow\quad 2b=na_n\ \Rightarrow\ b=\frac{n}{2}a_n$$

Therefore, (c) is the answer

6. We have

$$(1+x)^m(1-x)^n=\left[1+mx+\frac{m(m-1)}{2}x^2+\ldots\right]\left[1-nx+\frac{n(n-1)}{2}x^2-\ldots\right]$$

$$=1+(m-n)x+\left[\frac{m(m-1)}{2}+\frac{n(n-1)}{2}-mn\right]x^2+\ldots$$

term containing power of $x\geq 3$.

Now, $\qquad m-n=3 \qquad\ldots(1)$

(coefficient of $x=3$ given)

and $\frac{1}{2}m(m-1)+\frac{1}{2}n(n-1)-mn=-6$

or $\quad m(m-1)+n(n-1)-2mn=-12$

$\Rightarrow\quad m^2-m+n^2-n-2mn=-12$

$\Rightarrow\quad (m-n)^2-(m+n)=-12$

$\Rightarrow\quad m+n=9+12=21 \qquad\ldots(2)$

Solving (1) and (2)

$m=12$. therefore (c) is the answer.

7. $$\binom{n}{r}+2\binom{n}{r-1}+\binom{n}{r-2}=\left[\binom{n}{r}+\binom{n}{r-1}\right]+\left[\binom{n}{r-1}+\binom{n}{r-2}\right]$$

$$=\binom{n+1}{r}+\binom{n+1}{r-1}=\binom{n+2}{r}$$

$[\because {}^nC_r+{}^nC_{r-1}={}^{n+1}C_r$ (formula)]

So, answer is (d)

8. Given $T_5+T_6=0$

$$\Rightarrow\quad {}^nC_4a^{n-4}b^4-{}^nC_5a^{n-5}b^5=0$$

$$\Rightarrow\quad {}^nC_4a^{n-4}b^4={}^nC_5a^{n-5}b^5$$

$$\Rightarrow\quad \frac{a}{b}=\frac{{}^nC_5}{{}^nC_4}=\frac{n-4}{5}$$

Therefore, (b) is the answer

9. According to given condition, $T_n={}^nC_3$

and $\qquad T_{n+1}-T_n=21$

$$\Rightarrow\quad {}^{n+1}C_3-{}^nC_3=21$$

$\Rightarrow \quad \frac{1}{6}(n+1)(n)(n-1) - \frac{1}{6}n(n-1)(n-2) = 21$

$\Rightarrow \quad \frac{n(n-1)}{6}[(n+1)-(n-2)] = 21$

$\Rightarrow \quad \frac{n(n-1)\cdot 3}{6} = 21$

$\Rightarrow \quad n(n-1) = 42$

$\Rightarrow n = 7.$

Therefore, (b) is the answer.

10. $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$ is the coefficient of x^m in the expansion of $(1+x)^{10}(x-1)^{20}$

$\Rightarrow \sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$ is the coefficient of x^m in the expansion of $(1+x)^{30}$

i.e., $\quad \sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i} = {}^{30}C_m = \binom{30}{m} \quad \ldots(1)$

and we know $\binom{n}{r}$ is maximum, when

$$\binom{n}{r}_{\max} = \begin{cases} r = \frac{n}{2}, & n \in \text{even} \\ r = \frac{n\pm 1}{2}, & n \in \text{odd} \end{cases}$$

$\therefore \binom{30}{m}$ is maximum when $m = 15$.

11. Here, coefficient of t^{24} in $\{(1+t^2)^{12}(1+t^{12})(1+t^{24})\}$

$\Rightarrow$ coefficient of t^{24} in

$\{(1+t^2)^{12}.(1+t^{12}+t^{24}+t^{36})\}$

$\Rightarrow$ coefficient of t^{24} in

$\{(1+t^2)^{12} + t^{12}(1+t^2)^{12} + t^{24}(1+t^2)^{12}\};$

(neglecting $t^{36}(1+t^2)^{12}$)

$\Rightarrow$ coefficient of t^{24} is $({}^{12}C_{12} + {}^{12}C_6 + 1)$

$\Rightarrow$ coefficient of t^{24} is $(2 + {}^{12}C_6)$.

12. Here, ${}^{n-1}C_r = (k^2-3)\,{}^{n}C_{r+1}$

$\Rightarrow \quad {}^{n-1}C_r = (k^2-3).\frac{n}{r+1}\,{}^{n-1}C_r$

$\Rightarrow \quad k^2 - 3 = \frac{r+1}{n}$

(Since, $n \ge r \Rightarrow \frac{r-1}{n} \le 1$ and $n, r > 0$)

$\Rightarrow \quad 0 < k^2 - 3 \le 1$

or $\quad 3 < k^2 \le 4$

$\Rightarrow \quad k \in [-2, -\sqrt{3}) \cup (\sqrt{3}, 2]$

13. Let

$A = \binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots + \binom{30}{20}\binom{30}{30}$

or $A = {}^{30}C_0\,.{}^{30}C_{10} - {}^{30}C_1\,.{}^{30}C_{11} + {}^{30}C_2\,.{}^{30}C_{12} \ldots + {}^{30}C_{20}\,.{}^{30}C_{30}$

= coefficient of x^{20} in $(1+x)^{30}(1-x)^{30}$

= coefficient of x^{20} in $(1-x^2)^{30}$

= coefficient of x^{20} in $\sum_{r=0}^{30}(-1)^r\,{}^{30}C_r(x^2)^r$

$= (-1)^{10}\,{}^{30}C_{10} \quad$ {for coefficient of x^{20}, let $r = 10$}

Hence

$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots + \binom{30}{20}\binom{30}{30} = {}^{30}C_{10}$

E SUBJECTIVE QUESTIONS

1. We know

$\sum\sum_{0\le i<j\le n} 2\,C_iC_j = \sum_{i=0}^{n}\sum_{j=0}^{n}C_iC_j - \sum_{i=0}^{n}\sum_{j=0}^{n}C_iC_j$

$= \sum_{i=0}^{n}C_i\sum_{j=0}^{n}C_j - \sum_{i=0}^{n}C_i^2$

$= 2^n\,2^n - \{{}^{2n}C_n\}$

$= 2^{2n} - {}^{2n}C_n$

Thus $\sum\sum_{0\le i<j\le n} C_iC_j = \frac{2^{2n} - {}^{2n}C_n}{2} = 2^{2n-1} - \frac{(2n)!}{2\,(n!)^2}$

2. ${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \ldots + {}^{n+1}C_{n+1}s_n$

$= \sum_{r=1}^{n+1}{}^{n+1}C_r s_{r-1}$, where $s_n = 1 + q + q^2 + \ldots + q^n$

$= \frac{1-q^{n+1}}{1-q}$

$\therefore \sum_{r=1}^{n+1}{}^{n+1}C_r\left(\frac{1-q^r}{1-q}\right)$

$= \frac{1}{1-q}\left(\sum_{r=1}^{n+1}{}^{n+1}C_r - \sum_{r=1}^{n+1}{}^{n+1}C_r\,.q^r\right)$

$= \frac{1}{1-q}\{(1+1)^{n+1} - (1+q)^{n+1}\}$

$$= \frac{1}{1-q}\{2^{n+1} - (1+q)^{n+1}\} \qquad \dots(1)$$

Also $S_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \dots + \left(\frac{q+1}{2}\right)^n$

$$= \frac{1-\left(\frac{q+1}{2}\right)^{n+1}}{1-\left(\frac{q+1}{2}\right)} = \frac{2^{n+1}-(q+1)^{n-1}}{2^n(1-q)} \qquad \dots(2)$$

from (1) and (2)

$$^{n+1}C_1 + {}^{n+1}C_2 S_1 + {}^{n+1}C_3 S_2 + \dots + {}^{n+1}C_{n+1} S_n = 2^n S_n$$

3. $\sum_{r=0}^{n} (-1)^r\, {}^nC_r$

$$\left\{\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \dots \text{ up to } m \text{ terms}\right\}$$

$$= \sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{1}{2}\right)^r + \sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{3}{4}\right)^r + \sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{7}{8}\right)^r + \dots \text{ up to } m \text{ terms}$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \dots \text{ up to } m \text{ terms}$$

$$\left\{\text{using } \sum_{r=0}^{n} (-1)^r\, {}^nC_r x^r = (1-x)^n\right\}$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \dots \text{ up to } m \text{ terms}$$

$$= \left(\frac{1}{2}\right)^n \left\{\frac{1-\left(\frac{1}{2^n}\right)^m}{1-\frac{1}{2^n}}\right\} = \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

4. $C_0^2 - 2^2C_1 + 3^2C_2 \dots + (-1)^n (n+1)^2 C_n$

$$= \sum_{r=0}^{n} (-1)(r+1)^2\, {}^nC_r$$

$$= \sum_{r=0}^{n} (-1)^r (r^2+2r+1)\, {}^nC_r$$

$$= \sum_{r=0}^{n} (-1)^r r^2\, {}^nC_r + 2\sum_{r=0}^{n} (-1)^r r\, {}^nC_r + \sum_{r=0}^{n} (-1)^r .\, {}^nC_r$$

$$= \sum_{r=0}^{n} (-1)^r . r(r-1)\, {}^nC_r + 3.\sum_{r=0}^{n} (-1)^r . r\, {}^nC_r + \sum_{r=0}^{n} (-1)^r\, {}^nC_r$$

$$= \sum_{r=0}^{n} (-1)^r n(n-1)\, {}^{n-2}C_{r-2} + 3\sum_{r=1}^{n} (-1)^r n.\, {}^{n-1}C_{r-1} + \sum_{r=0}^{n} (-1)^r\, {}^nC_r$$

$$= n(n-1)\{{}^{n-2}C_0 - {}^{n-2}C_1 + {}^{n-2}C_2 \dots + (-1)^n\, {}^{n-2}C_{n-2}\} + 3n\{-{}^{n-1}C_0 + {}^{n-1}C_1 - {}^{n-1}C_2 \dots + (-1)^n\, {}^{n-1}C_{n-1}\} + \{{}^nC_0 - {}^nC_1 + {}^nC_2 \dots + (-1)^n\, {}^nC_n\}$$

$$= n(n-1).0 + 3n0 + 0$$

$$= 0$$

5. Let $y = (x-a)^m$, where m is a positive integer, $r \le m$

now $\frac{dy}{dx} = m(x-a)^{m-1}$

$$\Rightarrow \frac{d^2y}{dx^2} = m(m-1)(x-a)^{m-2}$$

$$\Rightarrow \frac{d^3y}{dx^3} = m(m-1)(m-2)(m-3)(x-a)^{m-4}$$

....

....

Differentiating r times, we get

$$\frac{d^ry}{dx^r} = m(m-1)\dots(m-r+1)(x-a)^{m-r}$$

$$= \frac{m!}{(m-r)!}(x-a)^{m-r}$$

$$= r!({}^mC_r)(x-a)^{m-r}$$

and for $r > m$, $\frac{d^ry}{dx^r} = 0$

Now, $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ (given)

Differentiating both sides n times w.r.t. x, we get

$$\sum_{r=1}^{2n} a_r (n!)^r C_n (x-2)^{r-n}$$

$$= \sum_{r=1}^{2n} b_r (n!)^r C_n (x-3)^{r-n}$$

Putting $x = 3$

$$\sum_{r=n}^{2n} a_r (n!)^r C_n = (b_n) n!$$

[∵ all the terms except first on R.H.S. become zero]

$\Rightarrow b_n = {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \dots + {}^{2n}C_n$

(∵ $a_r = 1 \ \forall r \ge n$)

$$= ({}^{n+1}C_{n+1} + {}^{n+1}C_n) + \dots + {}^{2n}C_n$$

$$= {}^{n+2}C_{n+1} + \dots + {}^{2n}C_n = \dots$$

$$= {}^{2n}C_{n+1} + {}^{2n}C_n = {}^{2n+1}C_{n+1}$$

6. Since n is an even positive integer, we can write $n=2m;\ m=1,2,3,\ldots$

with $\quad k=\frac{3n}{2}=\frac{3(2m)}{2}=3m$

Therefore, $\quad S=\sum_{r=1}^{3m}(-3)^{r-1}\,{}^{6m}C_{2r-1}$

i.e., $\quad S=(-3)^0\,{}^{6m}C_1+(-3)\,{}^{6m}C_3+\ldots+(-3)^{3m-1}\,{}^{6m}C_{3m-1}\quad \ldots(1)$

From the binomial expansion, we write

$$(1+x)^{6m}={}^{6m}C_0+{}^{6m}C_1x+{}^{6m}C_2x^2+\ldots+{}^{6m}C_{6m-1}x^{6m-1}+{}^{6m}C_{6m}x^{6m}\quad \ldots(2)$$

$$(1-x)^{6m}={}^{6m}C_0+{}^{6m}C_1(-x)+{}^{6m}C_2(-x)^2+\ldots+{}^{6m}C_{6m-1}(-x)^{6m-1}+{}^{6m}C_{6m}(-x)^{6m}\quad \ldots(3)$$

Subtracting equation (3) from equation (2), we get

$$(1+x)^{6m}-(1-x)^{6m}=2[{}^{6m}C_1x+{}^{6m}C_3x^3+{}^{6m}C_5x^5+\ldots+{}^{6m}C_{6m-1}x^{6m-1}]$$

$$\frac{(1+x)^{6m}-(1-x)^{6m}}{2x}={}^{6m}C_1+{}^{6m}C_3x^2+{}^{6m}C_5x^4+\ldots+{}^{6m}C_{6m-1}x^{6m-2}$$

Again let $x^2=y$

$$\Rightarrow\quad \frac{(1+\sqrt{y})^{6m}-(1-\sqrt{y})^{6m}}{2\sqrt{y}}={}^{6m}C_1+{}^{6m}C_3y+{}^{6m}C_5y^2+\ldots+{}^{6m}C_{6m-1}y^{3m-1}$$

For the required sum we have to put $y=-3$ in R.H.S.

Hence, $\quad S=\frac{[1+\sqrt{-3}]^{6m}-[1-\sqrt{-3}]^{6m}}{2\sqrt{-3}}$

$$=\frac{(1+i\sqrt{3})^{6m}-(1-\sqrt{-3})^{6m}}{2i\sqrt{3}}\quad \ldots(4)$$

Let $\quad z=1+i\sqrt{3}=r(\cos\theta+i\sin\theta)$

$\Rightarrow\quad r=\sqrt{|z|}=\sqrt{1+3}=2$ and $\theta=\pi/3$

Now, $z^{6m}=[r(\cos\theta+i\sin\theta)]^{6m}$

$=r^{6m}(\cos\theta+i\sin\theta)^{6m}$

$=r^{6m}\{\cos 6m\theta+i\sin 6m\theta\}$

Again $\quad \bar{z}=r(\cos\theta-\sin\theta)$

and $\quad (\bar{z})^{6m}=r^{6m}(\cos 6m\theta-i\sin 6m\theta)$

$\Rightarrow\quad z^{6m}-\bar{z}^{6m}=r^{6m}(2i\sin 6m\theta)\quad \ldots(5)$

Now, equation (4) becomes

$$S=\frac{z^{6m}-\bar{z}^{6m}}{2i\sqrt{3}}=\frac{r^{6m}(2i\sin 6m\theta)}{2i\sqrt{3}}$$

$$=\frac{2^{6m}\sin 6m\theta}{\sqrt{3}}$$

$=0$ as $m\in Z$, and $\theta=\pi/3$

7. $(1+x+x^2)^n=a_0+a_1x+\ldots+a_{2n}x^{2n}\quad \ldots(1)$

Replacing x by $1/x$, we obtain

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n=a_0-\frac{a_1}{x}+\frac{a_2}{x^2}-\frac{a_3}{x^3}+\ldots+\frac{a_{2n}}{x^{2n}}\quad \ldots(2)$$

Now, $a_0^2-a_1^2+a_2^2-a_3^2+\ldots+a_{2n}^2=$ coefficient of the term independent of x in

$$[a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}]\left[a_0-\frac{a_1}{x}+\frac{a_2}{x^2}-\ldots+\frac{a_{2n}}{x^{2n}}\right]$$

$=$ coefficient of the term independent of x in

$$(1+x+x^2)^n\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n$$

But $\quad$ R.H.S. $=(1+x+x^2)^n\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n$

$$=\frac{(1+x+x^2)^n(x^2-x+1)^n}{x^{2n}}$$

$$=\frac{[(x^2+1)^2-x^2]^n}{x^{2n}}$$

$$=\frac{(1+2x^2+x^4-x^2)^n}{x^{2n}}$$

$$=\frac{(1+x^2+x^4)^n}{x^{2n}}$$

Thus, $\quad a_0^2-a_1^2+a_2^2-a_3^2+\ldots a_{2n}^2$

$=$ coefficient of the term independent of x in $\frac{1}{x^{2n}}(1+x^2+x^4)^n$

$=$ coefficient of x^{2n} in $(1+x^2+x^4)^n$

$=$ coefficient of t^n in $(1+t+t^2)^n=a_n$

8. We have $\sum_{r=0}^{n}(-1)^r.\frac{{}^nC_r}{{}^{r+3}C_r}$

$$=\sum_{r=0}^{n}(-1)^r\frac{n!\,.3!}{(n-r)!\,(r+3)!}$$

$$=3!\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!\,(r+3)!}$$

$$=\frac{3!}{(n+1)(n+2)(n+3)}\sum_{r=0}^{n}\frac{(-1)^r.(n+3)!}{(n-r)!(r+3)!}$$

$$=\frac{3!}{(n+1)(n+2)(n+3)}\cdot\sum_{r=0}^{n}(-1)^r\,{}^{n+3}C_{r+3}$$

$$=\frac{3!(-1)^3}{(n+1)(n+2)(n+3)}\sum_{s=3}^{n+3}(-1)^s\,{}^{n+3}C_s$$

$$= \frac{-3!}{(n+1)(n+2)(n+3)}\left(\sum_{x=0}^{n+3}(-1)^x\,{}^{n+3}C_x\right) - {}^{n+3}C_0 + {}^{n+3}C_1 - {}^{n+3}C_2$$

$$= \frac{-3!}{(n+1)(n+2)(n+3)}\left\{0 - 1 + (n+3) - \frac{(n-3)(n+2)}{2!}\right\}$$

$$= \frac{-3!}{(n+1)(n+2)(n+3)}\cdot\frac{(n+2)(2-n-3)}{2}$$

$$= \frac{-3!}{2(n+3)}$$

9. This result can be proved by induction.

For $m=0$

$$\sum_{k=0}^{0}\frac{\binom{2n-k}{k}}{\binom{2n-k}{n}}\left(\frac{2n-4k+1}{2n-2k+1}\right)\cdot 2^{n-2k} = \frac{\binom{n}{0}}{\binom{2n}{n}}\cdot 2^n$$

$$\text{L.H.S.} = \frac{\binom{2n}{0}}{\binom{2n}{n}}\cdot\frac{2n+1}{2n+1}\cdot 2^{n-0} \qquad [\because k=0]$$

$$= \frac{\binom{n}{0}}{\binom{2n}{n}}\cdot 2^n = \text{R.H.S.} \qquad \ldots(1)$$

Assuming that the result holds for all non-negative integers $m < n$. That is, assume that

$$\sum_{k=0}^{m}\frac{\binom{2n-k}{k}}{\binom{2n-k}{n}}\left(\frac{2n-4k+1}{2n-2k+1}\right)2^{n-2k} = \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}}\cdot 2^{n-2m} \qquad \ldots(2)$$

For $m+1 \le n$, we have to show that

$$\sum_{k=0}^{m+1}\frac{\binom{2n-k}{k}}{\binom{2n-k}{n}}\left(\frac{2n-4k+1}{2n-2k+1}\right)\cdot 2^{n-2k} = \frac{\binom{n}{m+1}}{\binom{2n-2m-2}{n-m-1}}2^{n-2m-2} \qquad \ldots(3)$$

Now, L.H.S. of (3) can be written as

$$\sum_{k=0}^{m}\frac{\binom{2n-k}{k}}{\binom{2n-k}{n}}\left(\frac{2n-4k+1}{2n-2k+1}\right)\cdot 2^{n-2k}$$

$$+\frac{\binom{2n-(m+1)}{(m+1)}}{\binom{2n-(m+1)}{n}}\cdot\frac{\{2n-4(m+1)+1\}}{\{2n-2(m+1)+1\}}\times 2^{n-2(m+1)}$$

$$= \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}}2^{n-2m} + \frac{\binom{2n-m-1}{m+1}}{\binom{2n-m-1}{n}}\cdot\frac{(2n-4m-3)}{(2n-2m-1)}\cdot 2^{n-2m-2}$$

(by equation (2))

$$= \frac{n!}{m!(n-m)!}\times\frac{(n-m)!\,(n-m)!}{(2n-2m!)}\cdot 2^{n-2m}$$

$$+\frac{(2n-m-1)!}{(m+1)!\,(2n-2m-2)!}\cdot\frac{n!(n-m-1)!}{(2n-m-1)!}\cdot\frac{(2n-4m-3)}{(2n-2m-1)}\cdot 2^{n-2m-2}$$

$$= \frac{n!(n-m)!}{m!2(n-m)!}2^{n-2m} + \frac{n!(n-m-1)!(2n-4m-3)}{(m+1)!2(n-m-1)!(2n-2m-1)}\cdot 2^{n-2m-2}$$

$$= \frac{n!(n-m)!2^{n-2m}}{m!2^{n-m}(n-m)!}$$

$$+\frac{n!(n-m-1)!(2n-4m-3)\cdot 2^{n-2m-2}}{(m+1)!2^{n-m-1}\cdot(n-m-1)!(2n-2m-1)}$$

$$= \frac{n!}{m!2^m} + \frac{n!(2n-4m-3)}{(m+1)!2^{m+1}\cdot(2n-2m-1)}$$

$$= \frac{n!}{(m+1)!(n-m-1)!}\cdot\frac{(n-m-1)!(n-m-1)!2^{n-2m-2}}{(2n-2m-2)!}$$

$$\text{Again L.H.S. of (3)} = \frac{\binom{n}{m+1}}{\binom{2n-2m-2}{n-m-1}}\cdot 2^{n-2m-2} =$$

R.H.S. of (3)

Hence, the result hold for $m+1$.

By the principle of mathematical induction it holds for all non-negative integers $m < n$.

10. Let $S = \binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \ldots + \binom{m}{m} = \binom{n+1}{m+1}$ (given)

It is obvious that $n \ge m$

Imp. Note: This Question is based upon additive Loop.

Now, $S=\binom{m}{m}+\binom{m+1}{m}+\binom{m+2}{m}+\ldots+\binom{n}{m}$

$$=\left\{\binom{m+1}{m+1}+\binom{m+1}{m}\right\}+\binom{m+2}{m}+\ldots\binom{n}{m}$$

$$\left[\because \binom{m}{m}=1=\binom{m+1}{m+1}\right]$$

$$=\binom{m+2}{m+1}+\binom{m+2}{m}+\ldots+\binom{n}{m}$$

$$[\because {}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}]$$

$$=\binom{m+3}{m+1}+\ldots+\binom{n}{m}$$

$$=\ldots\ldots\ldots\ldots\ldots\ldots$$

$=\binom{n}{m+1}+\binom{n}{m}=\binom{n+1}{m+1}$ which is true. ...(1)

Again we have to prove that

$$\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+\ldots$$

$$+(n-m+1)\binom{m}{m}=\binom{n+2}{m+2}$$

$$S=\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+\ldots+(n-m+1)\binom{m}{m}$$

$$\left.\begin{array}{l}=\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+\ldots+\binom{m}{m}\\ \quad+\binom{n-1}{m}+\binom{n-2}{m}+\ldots+\binom{m}{m}\\ \quad\quad+\binom{n-2}{m}+\ldots+\binom{m}{m}\\ \quad\quad\quad\ldots\ldots\\ \quad\quad\quad+\binom{m}{m}\end{array}\right\} n-m+1 \text{ rows}$$

Now, sum of the first row is $\binom{n+1}{m+1}$,

sum of the second row is $\binom{n}{m+1}$,

sum of the third row is $\binom{n-1}{m+1}$,

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

sum of the last row is $\binom{m}{m}=\binom{m+1}{m+1}$.

Thus, $E=\binom{n+1}{m+1}+\binom{n}{m+1}+\binom{n-1}{m+1}+\ldots+\binom{m+1}{m+1}$

$$=\binom{n+1+1}{m+2}=\binom{n+2}{m+2}$$

from (1) replacing n by $n+1$ and m by $m+1$.

11. To show :

$$2^k.{}^nC_0.{}^nC_k-2^{k-1}.{}^nC_1.{}^{n-1}C_{k-1}$$

$$+2^{k-2}.{}^nC_2.{}^{n-2}C_{k-2}-\ldots(-1)^k\,{}^nC_k.{}^{n-k}C_0={}^nC_k$$

Taking L.H.S.

$$2^k.{}^nC_0.{}^nC_k-2^{k-1}.{}^nC_1.{}^{n-1}C_{k-1}$$

$$+\ldots+(-1)^k.{}^nC_k.{}^{n-k}C_0$$

$$=\sum_{r=0}^{k}(-1)^r\,2^{k-r}.{}^nC_r.{}^{n-r}C_{k-r}$$

$$=\sum_{r=0}^{k}(-1)^r\,2^{k-r}.\frac{n!}{r!(n-r)!}.\frac{(n-r)!}{(k-r)!(n-k)!}$$

$$=\sum_{r=0}^{k}(-1)^r.2^{k-r}.\frac{n!}{(n-k)!.k!}.\frac{k!}{r!(k-r)!}$$

$$=\sum_{r=0}^{k}(-1)^r.2^{k-r}\,{}^nC_k.{}^kC_r$$

$$=2^k.{}^nC_k\left\{\sum_{r=0}^{k}(-1)^r.\frac{1}{2^r}.{}^kC_r\right\}$$

$$=2^k.{}^nC_k\left(1-\frac{1}{2}\right)^k$$

$$={}^nC_k=\text{R.H.S.}$$

❑