

Find the following limits:

<p>(1) $\lim_{x \rightarrow 1} \frac{\sqrt{x+7} - \sqrt{3x+5}}{\sqrt{3x+5} - \sqrt{5x+3}}$</p> <p>[Ans: 1]</p>	<p>(2) $\lim_{x \rightarrow 1^+} \frac{\sqrt{x^2-1}}{\sqrt{x^3-1} + \sqrt{x^4-1}}$</p> <p>[Ans: $\sqrt{2} (2 - \sqrt{3})$]</p>
<p>(3) $\lim_{x \rightarrow 0} \frac{\sin ax - \sin bx}{\sin cx - \sin dx}$ [Ans: $\frac{a-b}{c-d}$]</p>	<p>(4) $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}}$ [Ans: 2]</p>
<p>(5) $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x - \cos x}{\frac{\pi}{4} - x}$ [Ans: $-\sqrt{2}$]</p>	<p>(6) $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 3x + \cos 3x}{x - \frac{\pi}{4}}$</p> <p>[Ans: $-3\sqrt{2}$]</p>
<p>(7) $\lim_{x \rightarrow \sqrt{3}} \frac{\tan^{-1} x - \frac{\pi}{3}}{x - \sqrt{3}}$ [Ans: $\frac{1}{4}$]</p>	<p>(8) $\lim_{x \rightarrow \frac{\pi}{6}} \frac{2 \sin x - 1}{\sqrt{3} \tan x - 1}$ [Ans: $\frac{3}{4}$]</p>
<p>(9) $\lim_{x \rightarrow -1^+} \frac{\sqrt{x^3+1} + \sqrt{x^5+1}}{\sqrt{x+1}}$</p> <p>[Ans: $\frac{1}{7} (\sqrt{35} + \sqrt{21})$]</p>	<p>(10) $\lim_{x \rightarrow 1} \frac{x + x^2 + x^3 + \dots + x^n - n}{x - 1}$</p> <p>[Ans: $\frac{n(n+1)}{2}$]</p>
<p>(11) $\lim_{x \rightarrow 3} \frac{x^4 - 8x^2 - x - 6}{2x^3 - 15x - 9}$ [Ans: $\frac{59}{39}$]</p>	<p>(12) $\lim_{x \rightarrow 2} \frac{x^6 - 64}{\frac{1}{x^3} - \frac{1}{2^3}}$ [Ans: 9×2^3]</p>
<p>(13) $\lim_{x \rightarrow 2} \frac{(3x+3)^{\frac{1}{4}} - (4x+1)^{\frac{1}{4}}}{x^3 - 8}$</p> <p>[Ans: $-\frac{1}{144\sqrt{3}}$]</p>	<p>(14) $\lim_{x \rightarrow 2} \frac{\sqrt[3]{x+6} - \sqrt[3]{2x+4}}{x^2 - 4}$</p> <p>[Ans: $-\frac{1}{48}$]</p>

Find the following limits:

$(15) \lim_{x \rightarrow \frac{1}{2}} \frac{\sin^{-1} x - \frac{\pi}{6}}{2x - 1} \quad \left[\text{Ans: } \frac{1}{\sqrt{3}} \right]$	$(16) \lim_{x \rightarrow a} \frac{\tan x - \tan a}{\tan^{-1} x - \tan^{-1} a} \quad \left[\text{Ans: } (1 + a^2) \sec^2 a \right]$
$(17) \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} \quad \left[\text{Ans: } \frac{a^2 - b^2}{c^2 - d^2} \right]$	$(18) \lim_{x \rightarrow a} \frac{(x^n - a^n) - na^{n-1}(x-a)}{(x-a)^2} \quad \left[\text{Ans: } \frac{n(n-1)}{2} a^{n-2} \right]$
$(19) \lim_{x \rightarrow 0} \frac{(1+mx)^n - (1+nx)^m}{x^2} \quad \left[\text{Ans: } \frac{mn(n-m)}{2} \right]$	$(20) \lim_{x \rightarrow 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n} \right), \quad m, n \in \mathbb{N} \quad \left[\text{Ans: } \frac{m-n}{2} \right]$
$(21) \lim_{x \rightarrow 1} \frac{x^{n+1} - (n+1)x + n}{(x-1)^2}, \quad n \in \mathbb{N} \quad \left[\text{Ans: } \frac{n(n+1)}{2} \right]$	$(22) \lim_{n \rightarrow \infty} \frac{1^2 + 3^2 + 5^2 + \dots + (2n-1)^2}{[1 + 3 + 5 + \dots + (2n-1)]^2} \quad \left[\text{Ans: } \frac{4}{3} \right]$
$(23) \lim_{x \rightarrow 0} \frac{2^{5x} - 2^{3x}}{\sin x} \quad \left[\text{Ans: } 2 \log 2 \right]$	$(24) \lim_{x \rightarrow 0} \left(\frac{1-x}{1+x} \right)^{\frac{1}{x}} \quad \left[\text{Ans: } e^{-2} \right]$
$(25) \lim_{x \rightarrow e} \frac{\log x - 1}{x - e} \quad \left[\text{Ans: } e^{-1} \right]$	$(26) \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} 3^{\frac{r}{n}} \quad \left[\text{Ans: } 2 \log_3 e \right]$
$(27) \lim_{x \rightarrow \infty} \sum \frac{1}{n(n+1)}, \quad n \in \mathbb{N} \quad \left[\text{Ans: } 1 \right]$	$(28) \lim_{x \rightarrow \infty} \frac{\sum n^3}{n \sum n^2}, \quad n \in \mathbb{N} \quad \left[\text{Ans: } \frac{3}{4} \right]$
$(29) \lim_{n \rightarrow \infty} \left(\sqrt{n^2 + n + 1} - n \right) \quad \left[\text{Ans: } \frac{1}{2} \right]$	$(30) \lim_{x \rightarrow 0} \frac{x \tan x}{e^x - 2 + e^{-x}} \quad \left[\text{Ans: } 1 \right]$

Find the following limits:

$(31) \lim_{x \rightarrow 0} \frac{e^{3x} - 2e^{2x} + 2^x}{x} \quad \left[\text{Ans: } \log \frac{2}{e} \right]$	$(32) \lim_{x \rightarrow 0} \frac{6^x - 3^x - 2^x + 1}{x^2} \quad \left[\text{Ans: } (\log 3)(\log 2) \right]$
$(33) \lim_{x \rightarrow 0} \frac{4\sqrt{x^4 + 1} - \sqrt{x^2 + 1}}{x^2} \quad \left[\text{Ans: } -\frac{1}{2} \right]$	$(34) \lim_{x \rightarrow 1} \frac{3\sqrt[3]{7 + x^3} - \sqrt{3 + x^2}}{-1} \quad \left[\text{Ans: } -\frac{1}{4} \right]$
$(35) \lim_{x \rightarrow \frac{\pi}{3}} \frac{\tan^3 x - 3 \tan x}{\cos \left(x + \frac{\pi}{6} \right)} \quad \left[\text{Ans: } -24 \right]$	$(36) \lim_{x \rightarrow 0} \frac{a^{x+1} - 2a^x - a + 2}{a^{x+1} - 3a^x - a + 3} \quad \left[\text{Ans: } \frac{a-2}{a-3} \right]$
$(37) \lim_{x \rightarrow a^+} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}} \quad \left[\text{Ans: } \frac{1}{\sqrt{2a}} \right]$	$(38) \lim_{x \rightarrow a^-} \frac{\sqrt{a^2 - x^2} + (a-x)}{\sqrt{a^3 - x^3} + \sqrt{a-x}}, \quad a > 0 \quad \left[\text{Ans: } \frac{\sqrt{2a}}{a\sqrt{3} + 1} \right]$
$(39) \lim_{x \rightarrow -8} \frac{\sqrt{1-x} - 3}{2 + 3\sqrt{x}} \quad \left[\text{Ans: } -2 \right]$	$(40) \lim_{x \rightarrow \sqrt{2}} \frac{x^4 - 4}{x^2 + 3x\sqrt{2} - 8} \quad \left[\text{Ans: } \frac{8}{5} \right]$
$(41) \lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} \quad \left[\text{Ans: } \frac{\pi}{180} \right]$	$(42) \lim_{x \rightarrow 0} \frac{(\sqrt{1+x^2} - 1) \sin^{-1} x}{(\tan^{-1} x)^3} \quad \left[\text{Ans: } \frac{1}{2} \right]$
$(43) \lim_{x \rightarrow 0} \frac{(\sin^{-1} x)^2}{1 - \sqrt{1-x^2}} \quad \left[\text{Ans: } 2 \right]$	$(44) \lim_{h \rightarrow 0} \frac{e^{\tan^{-1} h} - e^{\sin^{-1} h}}{\tan^{-1} h - \sin^{-1} h} \quad \left[\text{Ans: } 1 \right]$
$(45) \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} \quad \left[\text{Ans: } \frac{1}{8} \right]$	$(46) \lim_{x \rightarrow \infty} \frac{2\sqrt{x} + 3\sqrt[3]{x} + 5\sqrt[5]{x}}{\sqrt{3x-2} + \sqrt[3]{2x-3}} \quad \left[\text{Ans: } \frac{2}{\sqrt{3}} \right]$

Find the following limits:

(47) $\lim_{x \rightarrow 3} \frac{(x^3 + 27) \log(x - 2)}{x^2 - 9}$ [Ans: 9]	(48) $\lim_{x \rightarrow 0} (\cos x)^{\cot^2 x}$ [Ans: $e^{-\frac{1}{2}}$]
(49) $\lim_{x \rightarrow \frac{\pi}{4}} (\sin 2x)^{\tan^2 2x}$ [Ans: $e^{-\frac{1}{2}}$]	(50) $\lim_{x \rightarrow \infty} \left(\frac{3x - 2}{3x + 2} \right)^{3x}$ [Ans: e^{-4}]
(51) $\lim_{x \rightarrow 0} (1 - 4x)^{\frac{1 - x}{x}}$ [Ans: e^{-4}]	(52) $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^{\frac{\sin x}{x - \sin x}}$ [Ans: e^{-1}]
(53) $\lim_{n \rightarrow \infty} \frac{(n + 1) + (n + 2) + \dots + (n + n)}{n^2}$ [Ans: $\frac{3}{2}$]	(54) $\lim_{n \rightarrow 0} \frac{8^n - 4^n - 2^n + 1}{\sqrt{5} - \sqrt{4 + \cos n}}$ [Ans: $8\sqrt{5} (\log_e 2)^2$]
(55) $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{4r^2 - 1}$ [Ans: $\frac{1}{2}$]	(56) $\lim_{n \rightarrow \infty} \frac{3^{n+1} - 2^n}{3^n + 2^{n+1}}$ [Ans: 3]
(57) $\lim_{x \rightarrow \infty} \left[\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} \right]$ [Ans: $\frac{1}{2}$]	(58) $\lim_{x \rightarrow a} \frac{xe^{-x} - ae^{-a}}{x - a}$ [Ans: $(1 - a)e^{-a}$]
(59) $\lim_{n \rightarrow \infty} \cos \frac{x}{2} \cdot \cos \frac{x}{2^2} \cdot \dots \cdot \cos \frac{x}{2^n}$ [Ans: $\frac{\sin x}{x}$]	(60) $\lim_{\theta \rightarrow 0} \frac{1 - \cos\{1 - \cos(1 - \cos \theta)\}}{\theta^8}$ [Ans: $\frac{1}{128}$]
(61) $\lim_{x \rightarrow 0} \frac{35^x - 2 \cdot 7^x + 1}{x}$ [Ans: $\frac{35}{49}$]	(62) $\lim_{x \rightarrow 0} \frac{e^{\alpha x} - e^{\beta x}}{\sin \alpha x - \sin \beta x}$ [Ans: 1]

Find the following limits:

<p>(63) $\lim_{y \rightarrow a} \sin \frac{y - a}{2} \tan \frac{\pi y}{2a}$</p> <p style="text-align: right;">[Ans: $-\frac{a}{\pi}$]</p>	<p>(64) $\lim_{x \rightarrow a} \frac{\sin^{-1} x - \sin^{-1} a}{x - a}$</p> <p style="text-align: right;">[Ans: $\frac{1}{\sqrt{1 - a^2}}$]</p>
<p>(65) $\lim_{x \rightarrow 0} \frac{1 - \cos(1 - \cos x)}{x^4}$</p> <p style="text-align: right;">[Ans: $\frac{1}{8}$]</p>	<p>(66) $\lim_{x \rightarrow \sqrt{2}} \frac{4 \sec^{-1} x - \pi}{x - \sqrt{2}}$ [Ans: $2\sqrt{2}$]</p>
<p>(67) $\lim_{h \rightarrow 0} \frac{(a + h)^2 \sin(a + h) - a^2 \sin a}{h}$</p> <p style="text-align: right;">[Ans: $a^2 \cos a + 2a \sin a$]</p>	<p>(68) Prove that $\lim_{x \rightarrow 2} (3x + 2) \neq 5$</p>
	<p>(69) Prove that $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$</p>

Solve the following problems:

<p>(70) Express $\left\{ x \mid \frac{1}{ 2x - 3 } \leq \frac{1}{5}, x \in \mathbb{R} - \left\{ \frac{3}{2} \right\} \right\}$ as the complement of an interval.</p> <p style="text-align: right;">[Ans: $\mathbb{R} - (-1, 4)$]</p>
<p>(71) Express 1-neighbourhood of 0.1 in the interval and modulus forms.</p> <p style="text-align: right;">[Ans: $(-0.9, 1.1), x - 0.1 < 1$]</p>
<p>(72) Express the following set in the form of an interval and also in the form $N(p, \delta)$, $p \in \mathbb{R}, \delta > 0$. $\{ x \mid 0 \leq 3x + 1 < 2, x \in \mathbb{R} \}$.</p> <p style="text-align: right;">[Ans: $\left(-1, \frac{1}{3}\right), N\left(-\frac{1}{3}, \frac{2}{3}\right)$]</p>
<p>(73) Express $\left\{ x \mid \frac{1}{ 3x + 2 } \leq \frac{1}{5}, x \in \mathbb{R} - \left\{ -\frac{2}{3} \right\} \right\}$ as the complement of an interval.</p> <p style="text-align: right;">[Ans: $\mathbb{R} - \left(-\frac{7}{3}, 1\right)$]</p>

(74) Prove that the intersection of two δ -neighbourhoods of p is a δ -neighbourhood of p .

(75) Assuming that the intersection of two open intervals is either empty or it is an open interval, prove that the intersection of two neighbourhoods of $p \in \mathbb{R}$ is a neighbourhood of p .

(76) If $f(x) = \begin{cases} x, & x < -1 \\ 0, & x = 0 \\ x^2, & x > 1 \end{cases}$, then can we find $\lim_{x \rightarrow 0} f(x)$?

[Ans: No. \because f is not defined in any neighbourhood of zero.]

(77) Examine the continuity of f at $x = 0$ where $f(x) = \frac{|x|}{x}$, for $x \neq 0$, $f(0) = 1$.

[Ans: discontinuous]

(78) Examine the continuity of $f(x) = [x]$, $x \in \mathbb{R}$ at $n \in \mathbb{Z}$.

[Ans: discontinuous $\forall n \in \mathbb{Z}$]

(79) If for each $x, a \in D_f$, we have $|f(x) - f(a)| \leq |x - a|$, then prove that f is continuous at a .

(80) For $f(x) = x - [x]$ find $\lim_{x \rightarrow n^-} f(x)$ and $\lim_{x \rightarrow n^+} f(x)$ where $n \in \mathbb{Z}$ and hence examine the continuity of f at n .

[Ans: discontinuous]

(81) If $f(x) = \frac{1}{e^x - e^{-x}}$, $x \neq 0$, then prove that $\lim_{x \rightarrow 0} f(x)$ does not exist.

(82) Is it true that $\lim_{n \rightarrow 0} \frac{\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}}{n} = \log_e 2$?

[Ans: No. In $\binom{n}{r}$, $n \in \mathbb{N}$. \therefore n is not defined everywhere in any neighbourhood of 0.]

(83) Express 2.123123123..... as a vulgar fraction

[Ans: $\frac{707}{333}$]

<p>(84) If $f(x) = \frac{x}{\frac{1}{e^x} - 1}$, $x \neq 0$, find $\lim_{x \rightarrow 0} f(x)$. [Ans: 0]</p>
<p>(85) Find limit of the sequence $\frac{1 \cdot 5}{9 \cdot 13}, \frac{2 \cdot 7}{12 \cdot 17}, \frac{3 \cdot 9}{15 \cdot 21}, \dots$ [Ans: $\frac{1}{6}$]</p>
<p>(86) Find limit of the sequence $\frac{1^3}{1 \cdot 2 \cdot 3}, \frac{2^3}{4 \cdot 5 \cdot 5}, \frac{3^3}{7 \cdot 8 \cdot 7}, \dots$ [Ans: $\frac{1}{18}$]</p>
<p>(87) Obtain the maximum $\delta > 0$ such that $\forall x, x \in N^*(2, \delta) \Rightarrow f(x) \in N(13, 0.001)$, where $f(x) = 4x + 5, x \in R$. [Ans: 0.00025]</p>
<p>(88) For $f(x) = 2x, x \in \left\{1 + \frac{1}{n}, n \in N\right\}$, show that $\forall x \in N^*(1, \delta), x \in D_f \Rightarrow f(x) - 2 < \varepsilon$. What can be said about $\lim_{x \rightarrow 1} f(x)$? [Ans: limit does not exist as f is not defined everywhere in any neighbourhood of 1]</p>
<p>(89) Express $\left\{x \mid \left x + \frac{1}{x}\right < 4, x \in R, x \neq 0\right\}$ as the union of the intervals. [Ans: $(-2, -\sqrt{3}), (-2 + \sqrt{3}) \cup (2 - \sqrt{3}, 2 + \sqrt{3})$]</p>
<p>(90) Express $\left\{x \mid \left 1 + \frac{3}{x}\right > 2, x \in R - \{0\}\right\}$ as the union of the intervals. [Ans: $(-1, 0) \cup (0, 3)$]</p>
<p>(91) Express $\left\{x \mid \left x^2 - \frac{13}{2}\right < \frac{5}{2}, x \in R\right\}$ as the union of the intervals. [Ans: $(-3, -2) \cup (2, 3)$]</p>
<p>(92) Prove that $\lim_{x \rightarrow 4} f(x)$ where $f(x) = \sqrt{x} - [\sqrt{x}]$ does not exist.</p>

(93) Find the limit of the sequence $\sqrt{5}, \sqrt{5 \cdot \sqrt{5}}, \sqrt{5 \cdot \sqrt{5 \cdot \sqrt{5}}}, \dots$ [Ans: 5]

(94) Find minimum $m \in \mathbb{N}$ such that $n \geq m \Rightarrow f(n) \in (1, 0.01)$, $f(n) = \frac{3^n - 1}{3^n}$ [Ans: 5]

(95) If $f(x + y) = f(x) + f(y) \forall x, y \in \mathbb{R}$ and if f is continuous at $x = 0$, then prove that f is continuous at every $x \in \mathbb{R}$.

(96) If $f(x) = \frac{(3^{\sin x} - 1)^2}{x \log(1 + x)}$, $x \neq 0$ is continuous at $x = 0$, then find $f(0)$.
[Ans: $(\log 3)^2$]

(97) If $f(x) = \frac{\sin(a+1)x + \sin x}{x}$, $x < 0$
 $= c$, $x = 0$
 $= \frac{(x + bx^2)^{\frac{1}{2}} - x^{\frac{1}{2}}}{\frac{3}{bx^2}}$, $x > 0$ is continuous at $x = 0$, then find a, b, c .
 [Ans: $a = -\frac{3}{2}$, $b \neq 0$, $c = \frac{1}{2}$]

(98) If $f(x) = -2 \sin x$, $x \leq -\frac{\pi}{2}$
 $= A \sin x + B$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$
 $= \cos x$, $x \geq \frac{\pi}{2}$ is continuous on \mathbb{R} , then find A and B .
 [Ans: $A = -1$, $B = 1$]

Solve the following problems of infinite series:

(99) Find $\sum_{n=1}^{\infty} \frac{n^3 + n^2}{n!} x^n$ and hence obtain $\sum_{n=1}^{\infty} \frac{n^3 + n^2}{n!}$
 [Ans: $(x^3 + 4x^2 + 2x)e^x, 7e$]

Solve the following problems of infinite series:

(100) If $S_n = 1^2 + 2^2 + 3^2 + \dots + n^2$, then find $\sum_{n=1}^{\infty} \frac{S_n x^n}{n!}$.

[Ans : $\frac{1}{6}(2x^3 + 9x^2 + 6x)e^x$]

(101) Prove that $1 + \lim_{x \rightarrow 1} \left[\frac{1+x}{2!} + \frac{1+x+x^2}{3!} + \dots \infty \right] = e$

(102) Prove that $n! > \left(\frac{n}{e}\right)^n$

(103) Prove that $\frac{1 + \frac{1}{2!} + \frac{1}{4!} + \dots \infty}{1 + \frac{1}{3!} + \frac{1}{5!} + \dots \infty} = 1 + \frac{2}{e^2 - 1}$

(104) Find $\sum_{n=1}^{\infty} \frac{2(n+1)}{(2n+1)!}$

[Ans: $e - 2$]

(105) Find $\frac{7}{3 \cdot 4 \cdot 5} + \frac{9}{5 \cdot 6 \cdot 7} + \frac{11}{7 \cdot 8 \cdot 9} + \dots \infty$

[Ans : $3 \log 2 - \frac{11}{6}$]

(106) Prove that $\sum_{n=1}^{\infty} \frac{\log x \cdot 2n - 1}{(2n-1)!} = \frac{x^2 - 1}{2x}$

(107) Find $\sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n - 1}{(x+1)^n}$

[Ans: $\log x$]

(108) Prove that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2x+1)^{(2n-1)}} = \log \sqrt{\frac{1+x}{x}}$

(109) Find approximate value of $(131)^{\frac{1}{3}}$ correct to four decimal places.

[Ans: 5.0788]

(110) If n is very large, prove that $\left(1 + \frac{1}{n}\right)^n = e \left(1 - \frac{1}{2n} + \frac{11}{24n^2}\right)$