

Solve all problems vectorially:

(1) Obtain the unit vectors perpendicular to each of $\vec{x} = (1, 2, -1)$ and $\vec{y} = (1, 0, 2)$.

$$\left[\text{Ans: } \pm \left(\frac{4}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{-2}{\sqrt{29}} \right) \right]$$

(2) If α is the angle between two unit vectors \vec{a} and \vec{b} , then prove that $|\vec{a} - \vec{b} \cos \alpha| = \sin \alpha$.

(3) If a vector \vec{r} makes with X-axis and Y-axis angles of measures 45° and 60° respectively, then find the measure of the angle which \vec{r} makes with Z-axis.

[Ans: 60° or 120°]

(4) If \vec{x} and \vec{y} are non-collinear vectors of \mathbb{R}^3 , then prove that \vec{x} , \vec{y} and $\vec{x} \times \vec{y}$ are non-coplanar.

(5) If the measure of angle between $\vec{x} = \vec{i} + \vec{j}$ and $\vec{y} = t\vec{i} - \vec{j}$ is $\frac{3\pi}{4}$, then find t .

[Ans: 0]

(6) Show that for any $a \in \mathbb{R}$, the directions $(2, 3, 5)$ and $(a, a+1, a+2)$ cannot be the same or opposite.

(7) If θ is a measure of angle between unit vectors \vec{a} and \vec{b} , prove that $\sin \frac{\theta}{2} = \frac{1}{2} |\vec{a} - \vec{b}|$.

(8) If \vec{x} , \vec{y} and \vec{z} are non-coplanar, prove that $\vec{x} + \vec{y}$, $\vec{y} + \vec{z}$ and $\vec{z} + \vec{x}$ are also non-coplanar.

- (9) Show that the vectors $(1, 2, 1)$, $(1, 1, 4)$ and $(1, 3, -2)$ are coplanar. Also express each of these vectors as a linear combination of the other two.

$$\left[\begin{array}{l} \text{Ans: } (1, 2, 1) = \frac{1}{2}(1, 1, 4) + \frac{1}{2}(1, 3, -2); \quad (1, 1, 4) = 2(1, 2, 1) - (1, 3, -2) \\ (1, 3, -2) = 2(1, 2, 1) - (1, 1, 4) \end{array} \right]$$

[Note: These vectors are collinear besides being coplanar. Hence, a y vector of R^3 which is not collinear with them cannot be expressed as a linear combination of these vectors even if it is coplanar with them.]

- (10) Show that $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ are non-coplanar vectors. Also express any vector (x, y, z) of R^3 as a linear combination of these vectors.

$$\left[\text{Ans: } (x, y, z) = \frac{x + y - z}{2}(1, 1, 0) + \frac{x - y + z}{2}(1, 0, 1) + \frac{y + z - x}{2}(0, 1, 1) \right]$$

- (11) Prove that an angle in a semi-circle is a right angle.

- (12) Prove that the three altitudes in a triangle are concurrent.

- (13) If $A - P - B$ and $\frac{AP}{PB} = \frac{m}{n}$, then prove that for any point O in space,

$$n(\vec{OA}) + m(\vec{OB}) = (m + n)\vec{OP}.$$

- (14) Prove that $A(1, 5, 6)$, $B(3, 1, 2)$ and $C(4, -1, 0)$ are collinear. Find also the ratio in which A divides \overline{BC} from B .

[Ans: $-2 : 3$]

- (15) Find in which ratio and at which point does the XY -plane divide \overline{AB} where A is $(2, -2, 1)$ and B is $(1, 4, -5)$.

$$\left[\text{Ans: } 1 : 5 \text{ from } A \text{ at } \left(\frac{11}{6}, -1, 0 \right) \right]$$

(16) If $A(0, -1, -1)$, $B(16, -3, -3)$ and $C(-8, -1, -2)$ are given points, then find the point $D(x, y, z)$ in space so that $\vec{AB} = \vec{CD}$.

[Ans: $(8, -3, -4)$]

(17) $A(0, -1, -4)$, $B(1, 2, 3)$ and $C(5, 4, -1)$ are given points. If D is the foot of perpendicular from A on \overline{BC} , find its position vector.

[Ans: $(3, 3, 1)$]

(18) If the position vectors A, B, C of triangle ABC are $\vec{a}, \vec{b}, \vec{c}$ respectively, then show that the area of triangle $ABC = \frac{1}{2} |(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a})|$.

(19) Find the volume of a prism having a vertex at origin O and having coterminal edges $\overline{OA}, \overline{OB}, \overline{OC}$, where A is $(4, 3, 1)$, B is $(3, 1, 2)$ and C is $(5, 2, 1)$.

[Ans: 10 cubic units]

(20) Find the volume of tetrahedron having vertices $V(1, 1, 3)$, $A(4, 3, 2)$, $B(5, 2, 7)$ and $C(6, 4, 8)$.

[Ans: $\frac{14}{3}$ cubic units]

(21) If the forces of magnitudes $\sqrt{2}$, 2 and $\sqrt{3}$ units are applied to a particle in the directions of vectors $(-1, 0, 1)$, $(1, 0, 1)$ and $(1, 1, -1)$ respectively, then find the magnitude and direction of the resultant force.

[Ans: $\sqrt{5}$, $\left(\cos^{-1} \sqrt{\frac{2}{5}}, \cos^{-1} \sqrt{\frac{1}{5}}, \cos^{-1} \sqrt{\frac{2}{5}} \right)$]

(22) A boat is sailing to the east with a speed of $10\sqrt{2}$ km/hr. A man on boat feels that the wind is blowing from the south-east with a speed of 5 km/hr. Find the true velocity of the wind.

[Ans: $5\sqrt{5}$ km/hr at an angle $\cos^{-1} \frac{3}{\sqrt{10}}$ with east towards north]

(23) A force of magnitude $2\sqrt{10}$ units is acting on a particle in the direction $3\vec{i} - \vec{j}$ and a force of magnitude $3\sqrt{13}$ units is acting on the same particle in the direction $2\vec{i} + 3\vec{j}$. Under the influence of these forces, the particle is displaced from $A(1, 2)$ to $B(6, 4)$. Find the work done. [Ans: 74 units]

(24) Prove that the diagonals of a rhombus bisect each other orthogonally.

(25) If a pair of medians of a triangle are equal, then show that the triangle is isosceles.

(26) Show that the perpendicular bisectors of sides of any triangle are concurrent.

(27) Prove that the diagonals of a rhombus are bisectors of its angles.

(28) If \vec{AD} is a bisector of $\angle BAC$ in triangle ABC and if $D \in BC$, then show that $\frac{BD}{DC} = \frac{AB}{AC}$.

(29) $ABCDEF$ is a regular hexagon. Prove that $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF} = 3\vec{AD}$.

(30) Show that centroid and in-centre of an equilateral triangle are the same. Find the in-centre of the triangle with vertices $(6, 4, 6)$, $(12, 4, 0)$ and $(4, 2, -2)$.

$$\left[\text{Ans: } \left(\frac{22}{3}, \frac{10}{3}, \frac{4}{3} \right) \right]$$

(31) If A is $(1, 2, 1)$ and B is $(4, -1, 2)$, then find $S(x, y, z)$ such that $2\vec{AB} = \vec{AS}$.

$$[\text{Ans: } (7, -4, 3)]$$

- (32) Let $A(1, 2, -1)$ and $B(3, 2, 2)$ be given points. Find in which ratios from A and at which points do the XY -, YZ - and ZX -planes divide \overline{AB} .

$$\left[\text{Ans: } 1:2, \left(\frac{5}{3}, 2, 0\right); -1:3, \left(0, 2, \frac{5}{2}\right); \overleftrightarrow{AB} \text{ is parallel to } ZX\text{-plane} \right]$$

- (33) Show that $(6, 0, 1)$, $(8, -3, 7)$ and $(2, -5, 10)$ can be the vertices of some rhombus. Find the co-ordinates of the fourth vertex of this rhombus.

$$[\text{Ans: } (0, -2, 4)]$$

- (34) Show that $(1, 2, 4)$, $(-1, 1, 1)$, $(6, 3, 8)$ and $(2, 1, 2)$ are the vertices of a trapezium. Find the area of this trapezium.

$$\left[\text{Ans: } \frac{3}{2}\sqrt{59} \right]$$

- (35) Find the area of the parallelogram $ABCD$ if $\overrightarrow{AC} = \bar{a}$ and $\overrightarrow{BD} = \bar{b}$.

$$\left[\text{Ans: } \frac{1}{2} |\bar{a} \times \bar{b}| \right]$$

- (36) Find the volume of a prism having a vertex at origin and having edges $\overrightarrow{OA} = 2\bar{i} + \bar{j} + \bar{k}$, $\overrightarrow{OB} = 3\bar{i} - \bar{j} + \bar{k}$ and $\overrightarrow{OC} = -\bar{i} + \bar{j} - \bar{k}$.

$$[\text{Ans: } 4 \text{ cubic units}]$$

- (37) Show that $(4, 5, 1)$, $(0, -1, -1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ cannot be the vertices of any tetrahedron.

- (38) Find the volume of the tetrahedron with vertices $(4, 5, 1)$, $(0, -1, -1)$, $(3, 9, 4)$ and $(1, 2, 3)$.

$$\left[\text{Ans: } \frac{28}{3} \text{ cubic units} \right]$$

- (39) A mechanical boat is rowing towards the north with speed of 8 km/hr. If wind blows from the east with the speed of 10 km/hr, find the resulting speed of the boat and also the direction of resulting motion of the boat.

$$\left[\text{Ans: } 2\sqrt{41} \text{ km/hr at an angle of } \pi - \cos^{-1}\left(\frac{5}{\sqrt{41}}\right) \text{ with east towards north} \right]$$

- (40) A river flows with a speed of 5 units. A person desires to cross the river in a direction perpendicular to its flow. Find in which direction should he swim if his speed is 8 units.

$$\left[\text{Ans: At an angle of } \pi - \cos^{-1}\left(\frac{5}{8}\right) \text{ with the direction of flow of the river} \right]$$

- (41) If speed of a particle is 5 units towards the east and $\sqrt{8}$ units towards the south-west, then find the resultant speed of the particle and its direction.

$$\left[\text{Ans: } \sqrt{13} \text{ units at an angle of } \cos^{-1}\frac{3}{\sqrt{13}} \text{ with east towards south} \right]$$

- (42) A boat speeds towards the north at $6\sqrt{2}$ units. A man on the boat feels that the wind is blowing from the south-east at 5 units. Find the true velocity of the wind.

$$\left[\text{Ans: } \sqrt{157} \text{ units at an angle of } \pi - \cos^{-1}\left(\frac{5}{\sqrt{314}}\right) \text{ with east towards north} \right]$$

- (43) A steamer moves to the north-east with a speed of 40 units. A passenger on the steamer feels the wind to be blowing from the north with $25\sqrt{2}$ units. Find the true velocity of the wind.

$$\left[\text{Ans: } 5\sqrt{34} \text{ units at an angle of } \cos^{-1}\frac{4}{\sqrt{17}} \text{ with east towards south} \right]$$

- (44) A particle is displaced from A(2, 1) to B(4, 2) when forces of magnitudes $4\sqrt{5}$ in the direction $2\hat{i} + \hat{j}$ and $6\sqrt{5}$ in the direction $\hat{i} - 2\hat{j}$ are applied. Find the work done.

$$[\text{Ans: } 20 \text{ units}]$$