A FILL IN THE BLANKS

1. Let $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda+1 & -2\lambda & \lambda-4 \\ \lambda-3 & \lambda+4 & 3\lambda \end{vmatrix}$

   be an identity in λ, where p, q, r, s and t are constants. Then, the value of t is.... (IIT 1981; 2M)

2. The solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ is....... (IIT 1981; 2M)

3. Given that $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, the other two roots are... and... (IIT 1983; 2M)

4. The value of the determinant $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ is ... (IIT 1988; 2M)

5. For positive number x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is...... (IIT 1993; 2M)

B TRUE / FALSE

1. The determinants $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ and $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ are not identically equal. (IIT 1983; 1M)

C OBJECTIVE QUESTIONS

➡ ***Only one option is correct :***

1. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with value 1. Let C be the subset of A consisting of all determinants with value -1. Then : (IIT 1981; 2M)

   (a) C is empty
   (b) B has as many elements as C
   (c) $A = B \cup C$
   (d) B has twice as many elements as C

2. The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$, if : (IIT 1997C; 2M)

   (a) x, y, z are in A.P. (b) x, y, z are in G.P.
   (c) x, y, z are in H.P. (d) xy, yz, zx are in A.P.

3. The parameter, on which the value of the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$$

   does not depend upon, is : (IIT 1997; 2M)

   (a) a (b) p
   (c) d (d) x

4. $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$= 0$

5. Let $\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$

Dividing and multiplying R_1, R_2, R_3 by $\log x, \log y, \log z$ respectively

$= \frac{1}{\log x \log y \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$

$= 0$

B TRUE / FALSE

1. Let, $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$

Applying $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$

$= \frac{1}{abc}\cdot abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$\therefore \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Hence. (False)

C OBJECTIVE (ONLY ONE OPTION)

1. Since, A is the determinant of order 3 with entries 0 or 1 only.

where B is the subset of A consisting of all determinants with value 1.

{We know by interchanging any two rows or columns among them self sign changes}

given that, C is the subsets having determinant with value -1.

$\therefore B$ has as many elements as C.

2. Here, $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$

$C_1 \to C_1 - (pC_2 + C_3)$

$\Rightarrow \begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(xp^2+yp+yp+z) & xp+y & yp+z \end{vmatrix} = 0$

$\Rightarrow \quad -(xp^2+2yp+z)(xz-y^2) = 0$

$\therefore$ either $xp^2+2yp+z=0$ or $y^2 = xz$

$\Rightarrow$ x, y, z are in G.P.

Therefore (b) is the answer.

3. Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$

Applying $C_1 \to C_1 + C_3$

$\Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ \cos(p-d)x+\cos(p+d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x+\sin(p+d)x & \sin px & \sin(p+d)x \end{vmatrix}$

$\Rightarrow \Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ 2\cos px\cos dx & \cos px & \cos(p+d)x \\ 2\sin px\cos dx & \sin px & \sin(p+d)x \end{vmatrix}$

$C_1 \to C_1 - 2\cos dx\, C_2$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2-2a\cos dx & a & a^2 \\ 0 & \cos px & \cos(p+d)x \\ 0 & \sin px & \sin(p+d)x \end{vmatrix}$$

Expanding along C_1, we get

$\Rightarrow \Delta = (1+a^2-2a\cos dx)[\sin(p+d)x\cos px$

$-\sin px\cos(p+d)x]$

$\Rightarrow \Delta = (1+a^2-2a\cos dx)[\sin\{(p+d)x-px\}]$

$\Rightarrow \Delta = (1+a^2-2a\cos dx)[\sin dx]$

which is independent of p. Therefore, (b) is the answer.

4. $$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$$

Imp. Note : Observe in R_1 that $a_{11}+a_{12}=a_{13}$ Check this trend in R_2 and R_3

apply $R_3 \to R_3-(R_1+R_2)$

$$= \begin{vmatrix} 1 & x & 0 \\ 2x & x(x-1) & 0 \\ 3x(x-1) & x(x-1)(x-2) & 0 \end{vmatrix} = 0$$

$\therefore f(x)=0 \Rightarrow f(100)=0$. Therefore, (a) is the answer.

5. As the given system has non-zero solution

$$\Rightarrow 0 = \begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+k & k-1 & -1 \\ 1+k & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix}$$

apply $C_1 \to C_1 - C_2, C_2 \to C_2 + C_3$

$$= -\begin{vmatrix} 1+k & -(k+1) \\ 1+k & -2 \end{vmatrix} = 2(k+1)-(k+1)^2$$

$=(k+1)(2-k-1)=(k+1)(-k+1)$

$\Rightarrow$ $k=\pm 1$

Therefore, (d) is the answer.

Imp. Note : There is a golden rule in determinant that n one's $\Rightarrow$ $(n-1)$ zero's or n (constant) $\Rightarrow$ $(n-1)$ zero's for all constant should be in a single row or a single column.

6. $$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} \sin x+2\cos x & \cos x & \cos x \\ \sin x+2\cos x & \sin x & \cos x \\ \sin x+2\cos x & \cos x & \sin x \end{vmatrix}$$

$$= (2\cos x+\sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\Rightarrow \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x-\cos x & 0 \\ 0 & 0 & \sin x-\cos x \end{vmatrix} = 0$$

$\Rightarrow$ $(2\cos x+\sin x)(\sin x-\cos x)^2=0$

$\Rightarrow$ $2\cos x+\sin x=0$, or $\sin x-\cos x=0$

$\Rightarrow$ $2\cos x=-\sin x$, or $\sin x=\cos x$

$\Rightarrow \cot x=-1/2$ gives no solution in $-\frac{\pi}{4}\le x\le\frac{\pi}{4}$ and $\sin x=\cos x$

$\Rightarrow$ $\tan x=1 \Rightarrow x=\pi/4$

Therefore (c) is the answer.

7. Given equations $(x+ay=0, az+y=0, ax+z=0)$ has infinite solutions.

$\therefore$ using cramer's Rule, its determinant $=0$

$$\Rightarrow \begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1+a^3=0$$

or $a=-1$

8. Since no solution, $\Delta=0$ and any one amongst $\Delta_x, \Delta_y, \Delta_z$ is non-zero.

where $$\Delta = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

and $$\Delta z = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & -4 \\ 1 & 1 & 4 \end{vmatrix} = 8 \ne 0$$

$\Rightarrow$ $\lambda=1$

D Objective (More Than One Option)

1. (b), (e)

Applying, $C_3 \to C_3 - (\alpha C_1 + C_2)$ in

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0, \text{ we get}$$

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & -(a\alpha^2 + 2b\alpha + c) \end{vmatrix} = 0$$

$\Rightarrow \quad -(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0$

$\Rightarrow \quad a\alpha^2 + 2b\alpha + c = 0$ or $b^2 = ac$.

$\therefore$ α is root of $ax^3 + 2bx + c$ or a, b, c are in G.P.

E Subjective Questions

1. If $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$; $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \qquad R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1, \text{ we get}$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= (a+b+c)\{-(c-b)^2 - (a-b)(a-c)\}$$

$$= -(a+b+c)\{a^2 + b^2 + c^2 - ab - bc - ca\}$$

$$= -\frac{1}{2}(a+b+c)\{2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca\}$$

$$= -\frac{1}{2}(a+b+c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$$

Which is always negative.

2. If $\Delta = \begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$,

$R_2 \to R_2 - (R_1 + R_3)$, we get

$$\Delta = \begin{vmatrix} x^2 + x & x+1 & x-2 \\ 4 & 0 & 0 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix} \text{ applying}$$

$R_1 \to R_1 + \frac{x^2}{4}R_2$ and $R_3 \to R_1 + \frac{x^2}{4}R_2$, we get

$$\Delta = \begin{vmatrix} x & x+1 & x-2 \\ -4 & 0 & 0 \\ 2x+3 & 2x-1 & 2x-1 \end{vmatrix} \text{ Applying } R_3 \to R_3 - 2R_1$$

$$= \begin{vmatrix} x+0 & x+1 & x-2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x & x \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$\Rightarrow \quad \Delta = Ax + B.$

where, $A = \begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$ and $B = \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

3. The given system of equations has atleast one solution if $\Delta \neq 0$

i.e.,

$$3x - y + 4z = 3$$
$$x + 2y - 3z = -2$$
$$6x + 5y + \lambda z = -3$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} \neq 0$$

$$\Delta = 3(2\lambda + 15) + 1(\lambda + 18) + 4(5 - 12)$$
$$= 7\lambda + 35 = 7(\lambda + 5)$$

for unique solution $\Delta \neq 0$

$\Rightarrow \quad \lambda \neq -5$

for $\quad \lambda = -5 \quad \Rightarrow \quad \Delta = 0$

Then, $\Delta_1 = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & -5 \end{vmatrix} = 0$

$\Delta_2 = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & -5 \end{vmatrix} = 0$

$\Delta_3 = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix} = 0$

$\therefore \quad \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Rightarrow$ Infinitly many solutions.

let $z = -k$ then equations become

$$3x - y = 3 - 4k$$
$$x + 2y = 3k - 2$$

on solving, we get

$$x = \frac{4 - 5k}{7},\ y = \frac{13k - 9}{7},\ z = k.$$

4. Since α is repeated root of $f(x) = 0$.

$\therefore f(x) = a(x - \alpha)^2 . a \in$ constant $(\neq 0)$

let $\phi(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

{To show $\phi(x)$ is divisible by $(x - \alpha)^2$, it is sufficient to show that $\phi(\alpha)$ and $\phi'(\alpha) = 0$}.

$\therefore \phi(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0,$

as R_1 and R_2 are identical

again,

$\phi'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$\phi'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0,$

as R_1 and R_3 are identical.

Thus, α is a repeated root of $\phi(x) = 0$.

Hence, $\phi(x)$ is divisible by $f(x)$.

5. Let $\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix},$

Applying $C_3 \to C_3 + C_2$

$\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$

(as, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$)

applying $C_2 \to C_2 + C_1$

$\Delta = \begin{vmatrix} {}^xC_r & {}^{x-1}C_{r+1} & {}^{x+1}C_{r-2} \\ {}^yC_r & {}^{y+1}C_{r-1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z-1}C_{r+2} \end{vmatrix}$

applying $C_3 \to C_3 + C_2$

$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$

$\therefore \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix}$

$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$

6. The system of equations has non-trivial solution if, $\Delta = 0$.

$\Rightarrow \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$

expanding along C_1, we get

$\Rightarrow \sin 3\theta \cdot (28 - 21) - \cos 2\theta(-7 - 7) + 2(-3 - 4) = 0$

$\Rightarrow 7\sin 3\theta + 14\cos 2\theta - 14 = 0$

$\Rightarrow \sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow 3\sin\theta - 4\sin^3\theta + 2(1 - 2\sin^2\theta) - 2 = 0$

$\Rightarrow \sin\theta(4\sin^2\theta - 4\sin\theta - 3) = 0$

$\Rightarrow \sin\theta(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\Rightarrow \sin\theta = 0, \sin\theta = 1/2$

{neglecting $\sin\theta = -3/2$}

$\Rightarrow \theta = n\pi, n\pi + (-1)^n \pi/6, n \in Z.$

7. Let $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2 - 3n \end{vmatrix}$

$$\therefore \sum_{a=1}^{n} \Delta_a = \begin{vmatrix} \sum_{a=1}^{n}(a-1) & n & 6 \\ \sum_{a=1}^{n}(a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^{n}(a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{n(n-1)(2n-1)}{6} & 2n^2 & (4n-2) \\ \frac{n^2(n-1)^2}{4} & 3n^3 & (3n^2-3n) \end{vmatrix}$$

$$= \frac{n^2(n-1)}{2} \begin{vmatrix} 1 & 1 & 6 \\ \frac{(2n-1)}{3} & 2n & (4n-2) \\ \frac{n(n-1)}{2} & 3n^2 & (3n^2-3n) \end{vmatrix}$$

$$= \frac{n^3(n-1)}{12} \begin{vmatrix} 1 & 1 & 6 \\ (2n-1) & 6n & 12n-6 \\ (n-1) & 6n & 6n-6 \end{vmatrix}$$

$C_3 \to C_3 - 6C_1$

$$= \frac{n^3(n-1)}{12} \begin{vmatrix} 1 & 1 & 0 \\ 2n-1 & 6n & 0 \\ n-1 & 6n & 0 \end{vmatrix} = 0$$

$\Rightarrow \sum_{a=1}^{n} \Delta_a = C$, $\{C = 0$ *i.e.*, constant$\}$

8. We know,

$$A28 = A \times 100 + 2 \times 10 + 8$$

$$3B9 = 3 \times 100 + B \times 10 + 9$$

and, $$62C = 6 \times 100 + 2 \times 10 + C$$

Since ; $A28, 3B9$ and $62C$ are divisible by K, therefore there exist positive integers m_1, m_2 and m_3 such that,

$$100 \times A + 10 \times 2 + 8 = m_1 K$$

$$100 \times 3 + 10 \times B + 9 = m_2 K$$

and, $$100 \times 6 + 10 \times 2 + C = m_3 K \quad \ldots (1)$$

$$\therefore \Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$$

Applying $R_2 \to 100R_1 + 10R_3 + R_2$

$$\Rightarrow \Delta = \begin{vmatrix} A & 3 & 6 \\ 100A + 2\times 10 & 100\times 3 + 10 \times B + 9 & 100 \times 6 + 10 \times 2 + C \\ 2 & B & 2 \end{vmatrix}$$

$$= \begin{vmatrix} A & 3 & 6 \\ A28 & 3B9 & 62C \\ 2 & B & 2 \end{vmatrix}, \quad \text{(using (1))}$$

$$= \begin{vmatrix} A & 3 & 6 \\ m_1K & m_2K & m_3K \\ 2 & B & 2 \end{vmatrix} = K \begin{vmatrix} A & 3 & 6 \\ m_1 & m_2 & m_3 \\ 2 & B & 2 \end{vmatrix}$$

$\therefore$ $\Delta = mK$. Hence determinant is divisible by K.

9. Let, $\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix}$$

$$= c\begin{vmatrix} a-p & q-b \\ a-p & 0 \end{vmatrix} + (r-c)\begin{vmatrix} p & b \\ a-p & q-b \end{vmatrix}$$

$$= -c(a-p)(q-b) + (r-c)\{p(q-b) - b(a-p)\}$$

$$= -c(a-p)(q-b) + p(r-c)(q-b) - b(r-c)(a-p).$$

$$\Delta = 0$$

$$\Rightarrow -c(a-p)(q-b) + p(r-c)(q-b) - b(a-p)(r-c) = 0$$

$$\Rightarrow \frac{c}{r-c} + \frac{p}{p-a} + \frac{b}{q-b} = 0$$

{On dividing both sides by $(a-p)(q-b)(r-c)$}

$$\Rightarrow \frac{p}{p-a} + \frac{b}{q-b} + 1 + \frac{c}{r-c} + 1 = 2$$

$$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2.$$

10. $D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$ (given)

Taking $n!$, $(n+1)!$ and $(n+2)!$ common from R_1, R_2 and R_3 respectively,

$$D = n!(n+1)!(n+2)! \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 1 & (n+2) & (n+2)(n+3) \\ 1 & (n+3) & (n+3)(n+4) \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we get

$$D = n!(n+1)!(n+2)! \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 0 & 1 & 2n+4 \\ 0 & 1 & 2n+6 \end{vmatrix}$$

Expanding along C_1, we get

$$D = (n!)(n+1)!(n+2)![(2n+6) - (2n+4)]$$

$$D=(n!)(n+1)!(n+2)![2]$$

Divide both side by $(n!)^3$

$$\Rightarrow \quad \frac{D}{(n!)^3}=\frac{(n!)(n!)(n+1)(n!)(n+1)(n+2)2}{(n!)^3}$$

$$\Rightarrow \quad \frac{D}{(n!)^3}=2(n+1)(n+1)(n+2)$$

$$\Rightarrow \quad \frac{D}{(n!)^3}=2(n^3+4n^2+5n+2)=(2n^3+8n^2+10n+4)$$

$$=2n(n^2+4n+5)+4$$

$$\Rightarrow \quad \frac{D}{(n!)^3}-4=2n(n^2+4n+5)$$

which shows that $\left[\frac{D}{(n!)^3}-4\right]$ is divisible by n

11. Given, $\lambda x+(\sin\alpha)y+(\cos\alpha)z=0$

$x+(\cos\alpha)y+(\sin\alpha)z=0$

$-x+(\sin\alpha)y-(\cos\alpha)z=0$ has non-trivial solution.

$$\Rightarrow \quad \Delta=0$$

$$\Rightarrow \quad \begin{vmatrix}\lambda & \sin\alpha & \cos\alpha\\ 1 & \cos\alpha & \sin\alpha\\ -1 & \sin\alpha & -\cos\alpha\end{vmatrix}=0$$

$$\Rightarrow \quad \lambda(-\cos^2\alpha-\sin^2\alpha)-\sin\alpha(\cos\alpha-\sin\alpha)$$
$$+\cos\alpha(\sin\alpha+\cos\alpha)=0$$

$$\Rightarrow \quad -\lambda+\sin\alpha\cos\alpha+\sin\alpha\cos\alpha-\sin^2\alpha+\cos^2\alpha=0$$

$$\Rightarrow \quad \lambda=\cos 2\alpha+\sin 2\alpha$$

{we know, $-\sqrt{a^2+b^2}\le a\sin\theta+b\cos\theta\le\sqrt{a^2+b^2}$}

$$\therefore \quad -\sqrt{2}\le\lambda\le\sqrt{2} \quad \ldots(1)$$

Again when $\lambda=1$,

$$\cos 2\alpha+\sin 2\alpha=1$$

or $\quad \frac{1}{\sqrt{2}}\cos 2\alpha+\frac{1}{\sqrt{2}}\sin 2\alpha=\frac{1}{\sqrt{2}}$

$$\Rightarrow \quad \cos(2\alpha-\pi/4)=\cos\pi/4$$

$$\therefore \quad 2\alpha-\pi/4=2n\pi\pm\pi/4$$

$$\Rightarrow \quad 2\alpha=2n\pi-\pi/4+\pi/4,\ 2\alpha=2n\pi+\pi/4+\pi/4$$

$$\therefore \quad \alpha=n\pi \text{ or } n\pi+\pi/4.$$

12. $\Delta=\begin{vmatrix}\cos(A-P) & \cos(A-Q) & \cos(A-R)\\ \cos(B-P) & \cos(B-Q) & \cos(B-R)\\ \cos(C-P) & \cos(C-Q) & \cos(C-R)\end{vmatrix}$ (given)

$$\Rightarrow \Delta=\begin{vmatrix}\cos A\cos P+\sin A\sin P & \cos(A-Q) & \cos(A-R)\\ \cos B\cos P+\sin B\sin P & \cos(B-Q) & \cos(B-R)\\ \cos C\cos P+\sin C\sin P & \cos(C-Q) & \cos(C-R)\end{vmatrix}$$

$$\Rightarrow \Delta=\begin{vmatrix}\cos A\cos P & \cos(A-Q) & \cos(A-R)\\ \cos B\cos P & \cos(B-Q) & \cos(B-R)\\ \cos C\cos P & \cos(C-Q) & \cos(C-R)\end{vmatrix}+\begin{vmatrix}\sin A\sin P & \cos(A-Q) & \cos(A-R)\\ \sin B\sin P & \cos(B-Q) & \cos(B-R)\\ \sin C\sin P & \cos(C-Q) & \cos(C-R)\end{vmatrix}$$

$$\Rightarrow \Delta=\cos P\begin{vmatrix}\cos A & \cos(A-Q) & \cos(A-R)\\ \cos B & \cos(B-Q) & \cos(B-R)\\ \cos C & \cos(C-Q) & \cos(C-R)\end{vmatrix}+\sin P\begin{vmatrix}\sin A & \cos(A-Q) & \cos(A-R)\\ \sin B & \cos(B-Q) & \cos(B-R)\\ \sin C & \cos(C-Q) & \cos(C-R)\end{vmatrix}$$

apply $C_2\to C_2-C_1\cos Q$, $C_3\to C_3-C_1\cos R$ in first determinant and $C_2\to C_2-C_1\sin Q$, and $C_3\to C_3-C_1\sin R$ in second.

$$\Delta=\cos P\begin{vmatrix}\cos A & \cos A\cos Q+\sin A\sin Q-\cos A\cos Q & \cos A\cos R+\sin A\sin R-\cos A\cos R\\ \cos B & \cos B\cos Q+\sin B\sin Q-\cos B\cos Q & \cos B\cos R+\sin B\sin R-\cos B\cos R\\ \cos C & \cos C\cos Q+\sin C\sin Q-\cos C\cos Q & \cos C\cos R+\sin C\sin R-\cos C\cos R\end{vmatrix}$$

$$+\sin P\begin{vmatrix}\sin A & \cos A\cos Q+\sin A\sin Q-\sin A\sin Q & \cos A\cos R+\sin A\sin R-\sin A\sin R\\ \sin B & \cos B\cos Q+\sin B\sin Q-\sin B\sin Q & \cos B\cos R+\sin B\sin R-\sin B\sin R\\ \sin C & \cos C\cos Q+\sin C\sin Q-\sin C\sin Q & \cos C\cos R+\sin C\sin R-\sin C\sin R\end{vmatrix}$$

$$\Rightarrow \Delta=\cos P\begin{vmatrix}\cos A & \sin A\sin Q & \sin A\sin R\\ \cos B & \sin B\sin Q & \sin B\sin R\\ \cos C & \sin C\sin Q & \sin C\sin R\end{vmatrix}+\sin P\begin{vmatrix}\sin A & \cos A\cos Q & \cos A\cos R\\ \sin B & \cos B\cos Q & \cos B\cos R\\ \sin C & \cos C\cos Q & \cos C\cos R\end{vmatrix}$$

$$\Delta=\cos P\sin Q\sin R\begin{vmatrix}\cos A & \sin A & \sin A\\ \cos B & \sin B & \sin B\\ \cos C & \sin C & \sin C\end{vmatrix}+\sin P\cos Q\cos R\begin{vmatrix}\sin A & \cos A & \cos A\\ \sin B & \cos B & \cos B\\ \sin C & \cos C & \cos C\end{vmatrix}$$

$$\Delta=0+0=0$$

13. Let $a>0$, $d>0$

and let

$$\Delta=\begin{vmatrix}\frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)}\\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)}\\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)}\end{vmatrix}$$

taking $\frac{1}{a(a+d)(a+2d)}$ common from

$R_1 \cdot \frac{1}{(a+d)(a+2d)(a+3d)}$

from $R_2 \cdot \frac{1}{(a+2d)(a+3d)(a+4d)}$ from R_3

$$=\frac{1}{a(a+d)^2(a-2d)^3(a+3d)^2(a+4d)}\begin{vmatrix}(a+d)(a+2d) & (a+2d) & a\\ (a+2d)(a+3d) & (a+3d) & (a+d)\\ (a+3d)(a+4d) & (a+4d) & (a+2d)\end{vmatrix}$$

$$\Delta=\frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\Delta'$$

where $\Delta'=\begin{vmatrix}(a+d)(a+2d) & (a+2d) & a\\ (a+2d)(a+3d) & (a+3d) & (a+d)\\ (a+3d)(a+4d) & (a+4d) & (a+2d)\end{vmatrix}$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$

$$=\begin{vmatrix}(a+d)(a+2d) & (a+2d) & a\\ (a+2d)(2d) & d & d\\ (a+3d)(2d) & d & d\end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$\Delta'=\begin{vmatrix}(a+d)(a+2d) & (a+2d) & a\\ (a+2d)2d & d & d\\ 2d^2 & 0 & 0\end{vmatrix}$$

Expanding along R_3, we get

$$\Delta'=2d^2\begin{vmatrix}a+2d & a\\ d & d\end{vmatrix}$$

$$\Delta'=(2d^2)(d)(a+2d-a)=4d^4$$

Therefore,

$$\Delta=\frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

14. since a, b, c are p^{th}, q^{th} and r^{th} terms of H.P.

$\Rightarrow$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, are in A.P.

or
$$\left.\begin{aligned}\frac{1}{a}&=A+(p-1)D\\ \frac{1}{b}&=A+(q-1)D\\ \frac{1}{c}&=A+(r-1)D\end{aligned}\right\} \quad \ldots(1)$$

$\therefore$
$$\Delta=\begin{vmatrix}bc & ca & ab\\ p & q & r\\ 1 & 1 & 1\end{vmatrix}$$

$$=abc\begin{vmatrix}\frac{1}{a} & \frac{1}{b} & \frac{1}{c}\\ p & q & r\\ 1 & 1 & 1\end{vmatrix}. \quad \text{(using (1))}$$

$$=abc\begin{vmatrix}A+(p-1)D & A+(q-1)D & A+(r-1)D\\ p & q & r\\ 1 & 1 & 1\end{vmatrix}$$

Applying $R_1 \to R_1 - (A-D)R_3 - DR_2$, we get

$$=abc\begin{vmatrix}0 & 0 & 0\\ p & q & r\\ 1 & 1 & 1\end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix}bc & ca & ab\\ p & q & r\\ 1 & 1 & 1\end{vmatrix}=0$$

15. Applying $R_3 \to R_3 - R_1 - 2R_2$, we get

$$f'(x)=\begin{vmatrix}2ax & 2ax-1 & 2ax+b+1\\ b & b+1 & -1\\ 0 & 0 & 1\end{vmatrix}$$

$$=\begin{vmatrix}2ax & 2ax-1\\ b & b+1\end{vmatrix}=\begin{vmatrix}2ax & -1\\ b & 1\end{vmatrix}$$

(Using $C_2 \to C_2 - C_1$)

$\Rightarrow$ $f'(x)=2ax+b$

Integrating, we get $f(x)=ax^2+bx+c$

where c is an arbitrary constant. Since f has maximum at $x=5/2$.

$\Rightarrow$ $f'(5/2)=0 \Rightarrow 5a+b=0$ ...(1)

Also, $f(0)=2 \Rightarrow c=2$

and $f(1)=-1 \Rightarrow a+b+c=-1$ ...(2)

Solving for (1) and (2) for a, b, we get

$a=1/4,\ b=-5/4$

Thus, $f(x)=\frac{1}{4}x^2-\frac{5}{4}x+2$

16. $$\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

Applying $R_2 \to R_2 + R_3$

$$= \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right)+\sin\left(2\theta-\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

Now, $\sin(\theta+\frac{2\pi}{3})+\sin(\theta-\frac{2\pi}{3})$

$$=2\sin\left(\frac{\theta+\frac{2\pi}{3}+\theta-\frac{2\pi}{3}}{2}\right)\cos\left(\frac{\theta+\frac{2\pi}{3}-\theta+\frac{2\pi}{3}}{2}\right)$$

$$=2\sin(\theta)\cdot\cos\frac{2\pi}{3}$$

$$=2\sin\theta\cdot\cos(\pi-\pi/3)$$

$$=-2\sin\theta\cos\pi/3=-\sin\theta$$

and $\cos(\theta+2\pi/3)+\cos(\theta-2\pi/3)$

$$=2\cos\left(\frac{\theta+\frac{2\pi}{3}+\theta-\frac{2\pi}{3}}{2}\right)\cos\left(\frac{\theta+\frac{2\pi}{3}-\theta+\frac{2\pi}{3}}{2}\right)$$

$$=2\cos\theta\cdot\cos(2\pi/3)$$

$$=2\cos\theta\left(-\frac{1}{2}\right)=-\cos\theta$$

and $\sin(2\theta+\frac{4\pi}{3})+\sin(2\theta-\frac{4\pi}{3})$

$$=2\sin\left(\frac{2\theta+\frac{4\pi}{3}+2\theta-\frac{4\pi}{3}}{2}\right)\times\cos\left(\frac{2\theta+\frac{4\pi}{3}-2\theta+\frac{4\pi}{3}}{2}\right)$$

$$=2\sin 2\theta.\cos 4\pi/3$$

$$=2\sin 2\theta.\cos(\pi+\pi/3)$$

$$=-2\sin 2\theta.\cos\pi/3=-\sin 2\theta$$

$$=\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin(\theta-2\pi/3) & \cos(\theta-2\pi/3) & \sin(2\theta-4\pi/3) \end{vmatrix}$$

$= 0$ (since R_1 and R_2 are proportional)

17. Given

$$\begin{vmatrix} ax-by-c & bx-ay & cx+a \\ bx-ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by-c \end{vmatrix}=0$$

$C_1 \to aC_1$

$$\begin{vmatrix} a^2x-aby-ac & bx+ay & cx+a \\ abx-a^2y & -ax+by-c & cy+b \\ acx+a^2 & cy+b & -ax-by+c \end{vmatrix}=0$$

Applying $C_1 \to C_1 + bC_2 + cC_3$

$$=\frac{1}{a}\begin{vmatrix} (a^2+b^2+c^2)x & ay+bx & cx+a \\ (a^2+b^2+c^2)y & by-c-ax & b+cy \\ a^2+b^2+c^2 & b+cy & c-ax-by \end{vmatrix}$$

$$=\frac{1}{a}\begin{vmatrix} x & ay+bx & cx+a \\ y & by-c-ax & b+cy \\ 1 & b-cy & c-ax-by \end{vmatrix}$$

$(\because a^2+b^2+c^2=1)$

$C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$

$$=\frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix}$$

$$=\frac{1}{ax}\begin{vmatrix} x^2 & axy & ax \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix}$$

$R_1 \to R_1 + yR_2 + R_3$

$$=\frac{1}{ax}\begin{vmatrix} x^2+y^2+1 & 0 & 0 \\ y & c-ax & b \\ 1 & cy & -ax-by \end{vmatrix}$$

Expanding along R_1

$$=\frac{1}{ax}[(x^2+y^2+1)\{(-c-ax)(-ax-by)-b(cy)\}]$$

$$=\frac{1}{ax}[(x^2+y^2+1)(acx+bcy+a^2x^2+abxy-bcy)]$$

$$= \frac{1}{ax}[(x^2 + y^2 + 1)acx + a^2x^2 + abxy]$$

$$= \frac{1}{ax} \cdot [ax(x^2 + y^2 + 1)(c + ax + by)]$$

$$\Delta = (x^2 + y^2 + 1)(ax + by + c)$$

but $\Delta = 0$ (given)

$\Rightarrow (x^2 + y^2 + 1)(ax + by + c) = 0$

$\Rightarrow \quad ax + by + c = 0,$

which represents a straight line.

18. As, $M^T M = I$ and $|M| = 1$

$\Rightarrow \quad |M^T M| = |I|$

$\Rightarrow \quad |M^T M| = |M| \quad \{as\ |I| = 1 = |M|\}$

$\Rightarrow \quad |M^T||M| - |M| = 0$

$\Rightarrow \quad |M|(|M^T| - 1) = 0$

$\Rightarrow \quad |M| = 0$ or $|M^T| = 1$ {neglecting $|M| = 0$}

$\Rightarrow \quad |M^T| = 1 \quad \ldots(1)$

$\therefore \quad |M - I| = |M - I||M^T| = |MM^T - M^T|$

$$= |I - M^T| = -|M^T - I|$$

$$= -|M - I|$$

$\Rightarrow \quad |M - I| + |M - I| = 0$

$\Rightarrow \quad |M - I| = 0$