D OBJECTIVE QUESTIONS

➡ ***More than one options are correct :***

1. The function $f(x)=\int_{-1}^{x} t(e^t-1)(t-1)(t-2)^3(t-3)^5\,dt$ has a local minimum at x equals to : **(IIT 1999; 3M)**

   (a) 0 (b) 1

   (c) 2 (d) 3

2. If $f(x)$ is cubic polynomial which has local maximum at $x=-1$. If $f(2)=18$, $f(1)=-1$ and $f'(x)$ has local minimum at $x=0$, then : **(IIT 2006)**

   (a) the distance between $(-1, 2)$ and $(a, f(a))$ where $x=a$ is the point of local minima is $2\sqrt{5}$.

   (b) $f(x)$ is increasing for $x\in[1, 2\sqrt{5}]$

   (c) $f(x)$ has local minima at $x=1$

   (d) the value of $f(0)=5$

3. If $f(x)=\begin{cases} e^x, & 0\le x\le 1 \\ 2-e^{x-1}, & 1<x\le 2 \\ x-e, & 2<x\le 3 \end{cases}$

   and $g(x)=\int_0^x f(t)\,dt$, $x\in[1,3]$ then : **(IIT 2006)**

   (a) $g(x)$ has local maxima at $x=1+\log_e 2$ and local minima at $x=e$.

   (b) $f(x)$ has local maxima at $x=1$ and local minima at $x=2$.

   (c) $g(x)$ has no local minima

   (d) $f(x)$ has no local maxima.

E SUBJECTIVE QUESTIONS

1. Let x and y be two real variables such that $x>0$ and $xy=1$. Find the minimum value of $x+y$. **(IIT 1981; 2M)**

2. A swimmer S is in the sea at a distance d km from the closest point A on a straight spere. The house of the swimmer is on the shore at a distance L km from A. He can swim at a speed of u km/hr and walk at a speed of v km/hr $(v>u)$. At what point on the shore should be land so that he reaches his house in the shortest possible time? **(IIT 1983; 2M)**

3. Find the coordinates of the point on the curve $y=\dfrac{x}{1+x^2}$ where the tangent to the curve has the greatest slope. **(IIT 1984)**

4. Let $f(x)=\sin^3 x+\lambda\sin^2 x$, $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$. Find the intervals in which λ should lie in order that $f(x)$ has exactly one minimum and exactly one maximum. **(IIT 1985; 5M)**

5. Let $A(p^2, -p)$ $B(q^2, q)$, $C(r^2, -r)$ be the vertices of the triangle ABC. A parallelogram $AFDE$ is drawn with vertices D, E and F on the line segments. BC, CA and AB respectively. Using calculus, show that maximum area of such a parallelogram is
$$\frac{1}{4}(p+q)(q+r)(p-r).$$
(IIT 1986; 5M)

6. Find the point on the curve
$$4x^2+a^2y^2=4a^2,\ 4<a^2<8$$
that is farthest from the point $(0, -2)$. **(IIT 1987)**

7. A point P is given on the circumference of a circle of radius r. Chords QR are parallel to the tangent at P Determine the maximum possible area of the triangle PQR. **(IIT 1990; 4M)**

8. A window of perimeter (including the base of the arch) is in the form of a rectangle surrounded by a semi-circle. The semi-circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass. The clear glass transmits three times as much light per square meter as the coloured glass does.
What is the ratio for the sides of the rectangle so that the window transmits the maximum light? **(IIT 1991; 4M)**

9. Let $f(x)=\begin{cases} -x^3+\dfrac{(b^3-b^2+b-1)}{(b^2+3b+2)} & 0\le x\le 1 \\ 2x-3, & 1\le x\le 3 \end{cases}$

   Find all possible real values of b such that $f(x)$ has the smallest value at $x=1$. **(IIT 1993; 5M)**

10. The circle $x^2+y^2=1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. Find the maximum area of the triangle QSR. **(IIT 1994; 5M)**

11. Let (h, k) be a fixed point, where $h>0$, $k>0$. A straight line passing through this point cuts the positive directions of the coordinate axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin. **(IIT 1995; 5M)**

12. Determine the points of maxima and minima of the function
$f(x)=\dfrac{1}{8}\ln x-bx+x^2$, $x>0$ where $b\ge 0$ is a constant. **(IIT 1996; 5M)**

13. Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c and d denote the lengths of the sides of the quadrilateral, prove that $2\le a^2+b^2+c^2+d^2\le 4$. **(IIT 1997)**

14. Suppose $f(x)$ is a function satisfying the following conditions

    (a) $f(0)=2$, $f(1)=1$

(b) has a minimum value at $x = \frac{5}{2}$ and

(c) $\forall x, f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax-b \end{vmatrix}$

where, a, b are some constants. Determine the constant a, b and the function $f(x)$. **(IIT 1998; 8M)**

15. A straight line L with negative slope passes through the point (8, 2) and cuts the positive coordinate axes at points P and Q. Find the absolute minimum value of $OP + OQ$, as L varies, where O is the origin. **(IIT 2002; 5M)**

16. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum. **(IIT 2003; 2M)**

17. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point $P(6, 8)$ to the circle and the chord of contact is maximum. **(IIT 2003; 2M)**

18. If $f(x)$ is twice differentiable function such that $f(a) = 0$, $f(b) = 2$, $f(c) = 1$, $f(d) = 2$, $f(e) = 0$, where $a < b < c < d < e$, then the minimum number of zero's of $g(x) = \{f'(x)\}^2 + f''(x) \cdot f(x)$ in the interval $[a, e]$ is ? **(IIT 2006)**

ANSWERS

A Fill in the Blanks

1. $\frac{1}{3}$ 2. $\frac{dr}{dt} = -\lambda$

B True / False

1. False

C Objective Questions (Only one option)

1. (b)   2. (b)   3. (c)   4. (b)   5. (b)   6. (d)   7. (d)
8. (a)   9. (d)   10. (d)

D Objective Questions (More than one option)

1. (b, d)   2. (b, c)   3. (a, b)

E Subjective Questions

1. 2   3. $x = 0, y = 0$ 4. $\lambda \in \left(\frac{-3}{2}, \frac{3}{2}\right)$ 6. (0, 2)   7. $\frac{3\sqrt{3}}{4} r^2$

8. $6 : 6 + \pi$ 9. $b \in (-2, -1) \cup [1, \infty)$ 10. $\frac{4\sqrt{3}}{9}$ 11. $2hk$ 12. $x = \alpha, x = \beta$

14. $a = \frac{1}{4}, b = -\frac{5}{4}, c = 2, f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$ 15. 18   16. (2, 1)   17. 5 units

18. 6

Solutions

A Fill in the Blanks

1. As, $A + B = \frac{\pi}{3}$ and we know product of terms is maximum when values are equal (when sum is given).

$\therefore$ $(\tan A \cdot \tan B)$ is maximum when $A = B = \frac{\pi}{6}$ i.e.,

$y = \tan \frac{\pi}{6} \cdot \tan \frac{\pi}{6} = \frac{1}{3}$.

2. Since, Rate of change of volume $\propto$ surface area

$\Rightarrow \quad \frac{dV}{dt} \propto S.A.$

or $\quad 4\pi r^2 \cdot \frac{dr}{dt} = -\lambda . 4\pi r^2$

or $\quad \frac{dr}{dt} = -\lambda$ is required differential equation.

B True / False

1. As, $\frac{\log_a x + \frac{1}{\log_a x}}{2} > 1$, using A.M. > G.M.

Here, equality holds only when $x = a$ which is not possible.

Hence, $\log_a x + \log_x a$ is greater than 2. (False)

C OBJECTIVE (ONLY ONE OPTION)

1. Since, $\max.(p, q) = \begin{cases} p, & p > q \\ q, & q > p \end{cases}$ and

$$\max.(p, q, r) = \begin{cases} p, & p \text{ is greatest} \\ q, & q \text{ is greatest} \\ r, & r \text{ is greatest} \end{cases}$$

$\therefore$ $\max.(p, q) < \max.(p, q, r)$ is false.

We know, $|p - q| = \begin{cases} p - q & p \geq q \\ q - p & q > p \end{cases}$

$$\therefore \frac{1}{2}\{p + q - |p - q|\} = \begin{cases} \frac{1}{2}(p + q - p + q), & p > q \\ \frac{1}{2}(p + q + p - q), & p < q \end{cases}$$

$$= \begin{cases} q, & p \geq q \\ p, & p < q \end{cases}$$

$$\Rightarrow \frac{1}{2}\{p + q - |p - q|\} = \min(p, q)$$

Hence, option (b) is correct.

2. $y = a \log x + bx^2 + x$ has extremum at $x = -1$ and $x = 2$.

$\therefore$ $\frac{dy}{dx} = 0$, at $x = -1$ and $x = 2$

$\Rightarrow$ $\frac{a}{x} + 2bx + 1 = 0$, at $x = -1$ and $x = 2$

$\therefore$ $-a - 2b + 1 = 0$ and $\frac{a}{2} + 4b + 1 = 0$

$\Rightarrow$ $a = 2$ and $b = -\frac{1}{2}$

3. $P(x) = a_0 + a_1 x^2 + a_2 x^4 + \ldots + a_n x^{2n}$

where ; $a_n > a_{n-1} > a_{n-2} \ldots > a_2 > a_1 > a_0 > 0$

$\Rightarrow P'(x) = 2a_1 x + 4a_2 x^3 + \ldots + 2na_n x^{2n-1}$

$= 2x\{a_1 + 2a_2 x^2 + \ldots + na_n x^{2n-2}\}$ ...(1)

where, $(a_1 + 2a_2 x^2 + 3a_3 x^4 + \ldots + na_n x^{2n-2}) > 0$

for all $x \in R$.

Thus, $\begin{cases} P'(x) > 0, & \text{when } x > 0 \\ P'(x) < 0, & \text{when } x < 0 \end{cases}$

i.e., $P'(x)$ changes sign from (–ve) to (+ve) at $x = 0$.

Hence, $P(x)$ attains minimum at $x = 0$

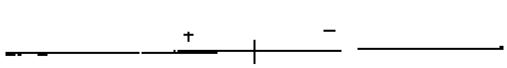
$\Rightarrow$ only one minimum at $x = 0$.

4. $f(x) = x^{25}(1 - x)^{75}, x \in [0, 1]$

$\Rightarrow$ $f'(x) = 25x^{24}(1 - x)^{75} - 75x^{25}(1 - x)^{74}$

$= 25x^{24}(1 - x)^{74}\{(1 - x) - 3x\}$

$= 25x^{24}(1 - x)^{74}(1 - 4x)$



$x = 1/4$

which shows $f'(x)$ is + ve for $x < 1/4$ and $f'(x)$ is – ve for $x > 1/4$.

$\therefore$ $f(x)$ attains maximum at $x = 1/4$.

Hence (b) is correct answer.

5. The maximum value of $f(x) = \cos x + \cos(\sqrt{2}x)$ is 2 which occurs at $x = 0$. Also, there is no value of x for which this value will be attained again.

Imp. note : This question can be solved by calculus also.

6. $$f(x) = \frac{x^2 - 1}{x^2 + 1} = 1 - \frac{2}{x^2 + 1}$$

$f(x)$ will be minimum when $\frac{2}{x^2 - 1}$ is maximum.

i.e., when $x^2 + 1$ is minimum

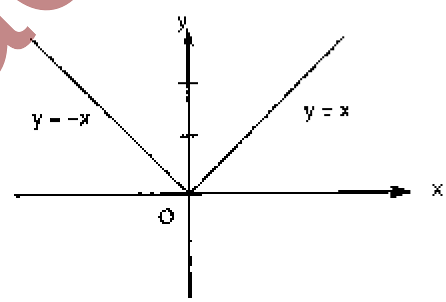
i.e., at $x = 0$.

$\therefore$ Minimum value of $f(x)$ is $f(0) = -1$

Therefore, (d) is the answer

7. It is clear from fig that at $x = 0$, $f(x)$ is not differentiable

$\Rightarrow$ neither maximum nor minimum.



Therefore, (d) is the answer.

8. $\cot\alpha_1 . \cot\alpha_2 ...... \cot\alpha_n = 1$

$\Rightarrow$ $\frac{\cos\alpha_1}{\sin\alpha_1} . \frac{\cos\alpha_2}{\sin\alpha_2} . \frac{\cos\alpha_3}{\sin\alpha_3} \ldots \frac{\cos\alpha_n}{\sin\alpha_n} = 1$

$\Rightarrow$ (Let) $\cos\alpha_1 . \cos\alpha_2 . \cos\alpha_3 \ldots \cos\alpha_n = k$ ...(1)

and $\sin\alpha_1 . \sin\alpha_2 . \sin\alpha_3 \ldots \sin\alpha_n = k$ ... (2)

again multiply (1) and (2), we get

$(\cos\alpha_1 . \cos\alpha_2 . \cos\alpha_3 \ldots \cos\alpha_n)$
$\times (\sin\alpha_1 . \sin\alpha_2 . \sin\alpha_3 \ldots \sin\alpha_n) = k^2$

$$k^2 = \frac{1}{2 \times 2 \times \ldots n \text{ times}} (2\sin\alpha_1 \cos\alpha_1)$$
$$(2\sin\alpha_2 \cos\alpha_2)\ldots(2\sin\alpha_n \cos\alpha_n)$$

$\Rightarrow$ $k^2 = \frac{1}{2^n}(\sin 2\alpha_1)(\sin 2\alpha_2) \ldots (\sin 2\alpha_n)$

$\leq \frac{1}{2^n}$ $\sin 2\alpha_i \leq 1$ for all $1 \leq i \leq n$

$\Rightarrow$ $k \leq \frac{1}{2^{n/2}}$

Therefore, (a) is the answer.

9. $f(x) = (1+b^2)x^2 + 2bx + 1$

It is a quadratic expression with coefficient of x^2 as $(1+b^2) > 0$

$\therefore$ $f(x)$ represents an upward parabola whose minimum value is $\frac{-D}{4a}$, D being discriminant.

$$\therefore \quad m(b) = \frac{-\{4b^2 - 4(1+b^2)\}}{4\cdot(1+b^2)}$$

$$\Rightarrow \quad m(b) = \frac{4}{1+b^2}$$

For Range of $m(b)$:

$$\frac{1}{1+b^2} > 0 \text{ also } b^2 \geq 0 \quad \Rightarrow \quad 1+b^2 \geq 1.$$

$$\Rightarrow \quad \frac{1}{1+b^2} \leq 1$$

Thus, $m(b) \in (0, 1]$

Hence (d) is the correct answer.

10. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$

Then $f(x)$ is minimum and $g(x)$ is maximum at $\left(x = \frac{-b}{a} \text{ and } f(x) = \frac{-D}{4a}\right)$ respectively.

$$\therefore \quad \text{min. } f(x) = -\frac{(4b^2 - 8c^2)}{4} = (2c^2 - b^2)$$

$$\text{and} \quad \text{max. } g(x) = -\frac{(4c^2 + 4b^2)}{4(-1)} = (b^2 + c^2)$$

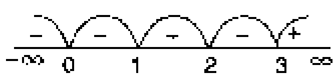
Now, min. $f(x) >$ max. $g(x)$

$$\Rightarrow \quad 2c^2 - b^2 > b^2 + c^2 \quad \text{or} \quad c^2 > 2b^2$$

$$\text{or} \quad |c| > \sqrt{2}\,|b|$$

D OBJECTIVE (MORE THAN ONE OPTION)

1. $$f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\,dt$$



$$f'(x) = \frac{d}{dx}\int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\,dt$$

$$= x(e^x - 1)(x-1)(x-2)^3(x-3)^5$$

$$\times 1 - x(e^x - 1)(x-1)(x-2)^3(x-3)^5 \times 0$$

$$\frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)\,dt = f(\psi(x))\psi'(x)$$

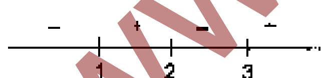
$$- f(\phi(x))\phi'(x) \text{ Formula}$$

For local minimum, $f'(x) = 0$

$$\Rightarrow \quad x = 0, 1, 2, 3.$$

Let $f'(x) = g(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5$

Using sign scheme rule,



This shows that $f(x)$ has a local minimum at $x = 1$ and $x = 3$ and maximum at $x = 2$. Therefore, (b) and (d) are the answer.

2. Since $f(x)$ has local maxima at $x = -1$ and $f'(x)$ has local minima at $x = 0$,

$$\therefore \quad f''(x) = \lambda x$$

$$f'(x) = \lambda\frac{x^2}{2} + c \qquad \{f'(-1) = 0\}$$

$$\Rightarrow \quad \frac{\lambda}{2} + c = 0$$

$$\Rightarrow \quad \lambda = -2c \qquad \ldots(i)$$

again, Integrating both sides we get

$$f(x) = \lambda\frac{x^3}{6} + cx + d \qquad \ldots(ii)$$

$$f(2) = \lambda\left(\frac{8}{6}\right) + 2c + d = 18$$

$$\text{and} \quad f(1) = \frac{\lambda}{6} + c + d = -1 \qquad \ldots(iii)$$

$\therefore$ using (i), (ii) and (iii) we get

$$f(x) = \frac{1}{4}(19x^3 - 57x + 34)$$

$$\therefore \quad f'(x) = \frac{1}{4}(57x^2 - 57)$$

$= \frac{57}{4}(x-1)(x+1)$, using number line rule

$\therefore$ $f(x)$ is increasing for $[1, 2\sqrt{5}]$

and $f(x)$ has local maximum at $x = -1$ and

$f(x)$ has local minimum at $x = 1$.

also, $f(0) = \frac{34}{4}$.

Hence, (b) and (c) are the correct answer.

3. Here,

$$f(x) = \begin{cases} e^x, & 0 \leq x \leq 1 \\ 2 - e^{x-1}, & 1 < x \leq 2 \\ x - e, & 2 < x \leq 3 \end{cases}$$

$$\text{and} \quad g(x) = \int_0^x f(t)\,dt$$

$$\Rightarrow \quad g'(x) = f(x)$$

where, $g'(x) = 0 \Rightarrow x = 1 + \log_e 2$ and $x = e$.